# Classical simulation of quantum circuits

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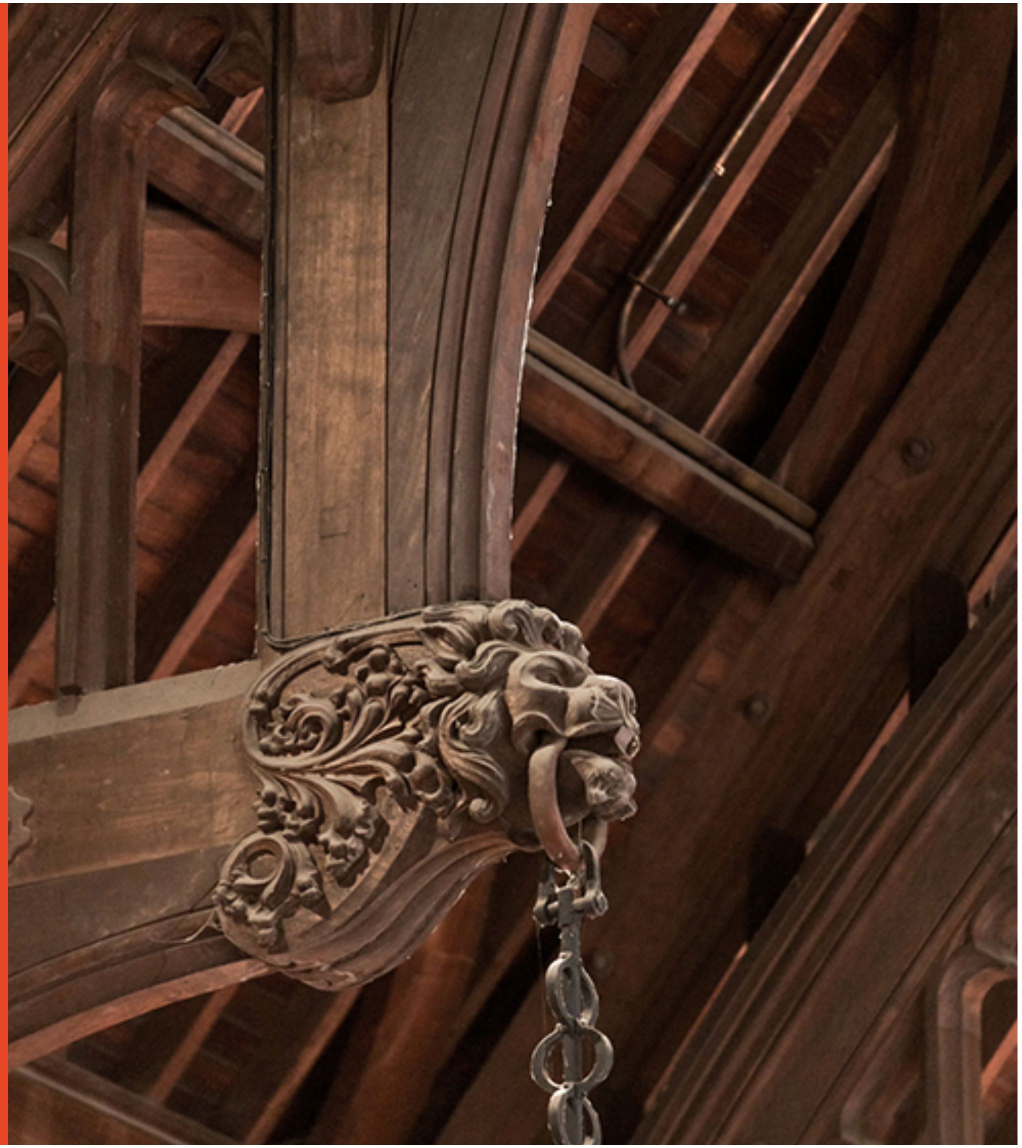
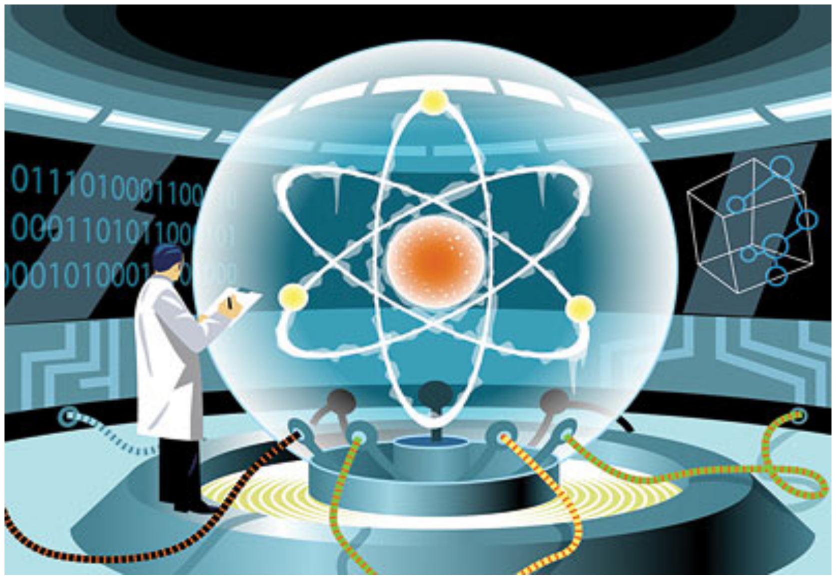


# The power of quantum computation



What makes quantum circuits/processes so hard to simulate?

- Exponentially large Hilbert space?
- Superposition of many ‘classical’ processes?
- Entanglement?

# Stabilizers and Gottesman-Knill

An efficiently simulatable subtheory

- A class of quantum circuits that can be efficiently simulated:
  - Initialize in a stabilizer state – includes entangled states (Bell, GHZ, cluster)
  - Unitaries in the Clifford group – including Paulis, CNOT, Hadamard, Phase gate
  - Measure in basis of stabilizer states

Aaronson and Gottesman, PRA (2004)

- Computation is somehow "classical"

- Stabilizer circuits can be viewed as supervening on a classical dynamical computation through the use of a nonnegative quasiprobability representation

# Quasiprobability representations

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# Quasiprobabilities

Quasiprobability representations: another way of describing quantum mech.

- Classical hidden variables on a phase space $\Lambda$

| | | |
|---|---|---|
| States | $\rho \longrightarrow W_\rho(\lambda)$ | Like a probability distribution |
| Unitaries | $U \longrightarrow W_U(\lambda\|\lambda')$ | Like a conditional probability |
| Measurements | $E \longrightarrow W(E\|\lambda)$ | Like a conditional probability |

- Real valued, normalized like probability distributions
- Born rule as you'd expect: $\mathrm{Tr}[EU\rho U^\dagger] = \sum_{\lambda,\lambda' \in \Lambda} W(E|\lambda) W_U(\lambda|\lambda') W_\rho(\lambda')$
- **But can go negative!**

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## Quasiprobabilities

Quasiprobability representations: another way of describing quantum mech.

- Classi…

Stat… …ion

Unit… …lity

Mea… …lity

- Re…
- Bo…
- **But …**

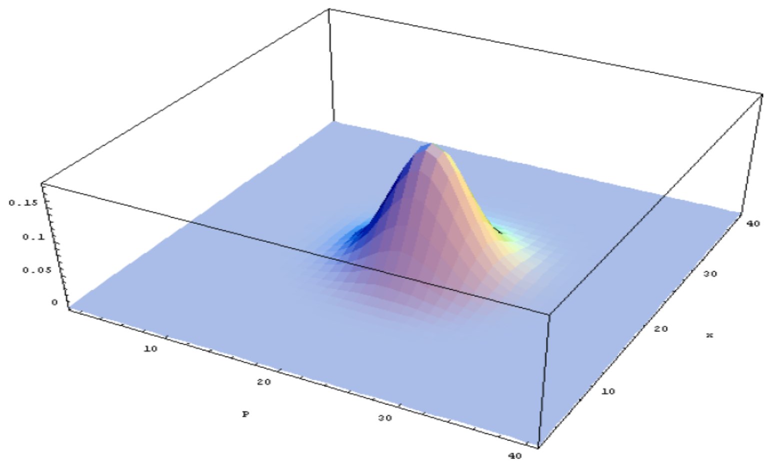
### Dual frames formalism

Two 'frames': $F(\lambda) : \lambda \in \Lambda$ and $G(\lambda) : \lambda \in \Lambda$

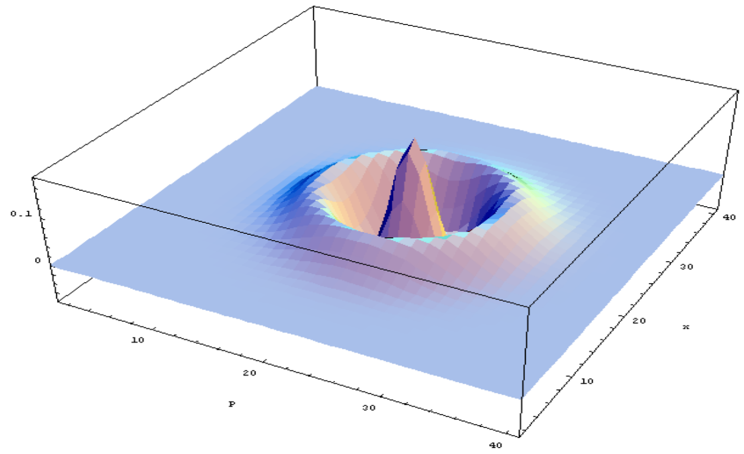
satisfying $A = \sum_{\lambda \in \Lambda} G(\lambda) \mathrm{Tr}[A F(\lambda)] \quad \forall\ A$

$$
\begin{aligned}
W_\rho(\lambda) &= \mathrm{Tr}[F(\lambda)\rho] \\
W_U(\lambda'|\lambda) &= \mathrm{Tr}(F(\lambda')UG(\lambda)U^\dagger) \\
W(E|\lambda) &= \mathrm{Tr}[EG(\lambda)]\,.
\end{aligned}
$$

# Negativity and nonclassicality



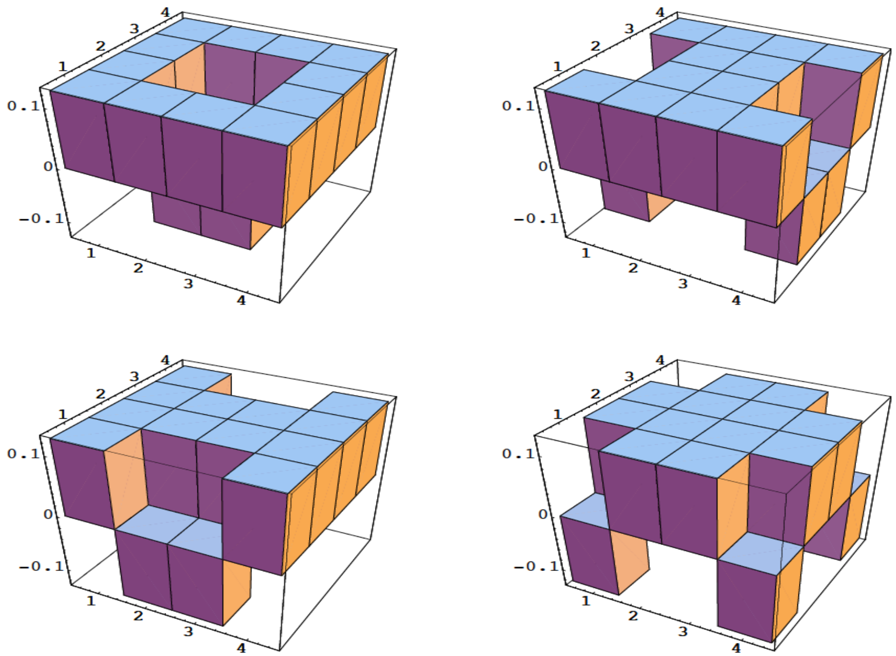

Classical




Quantum

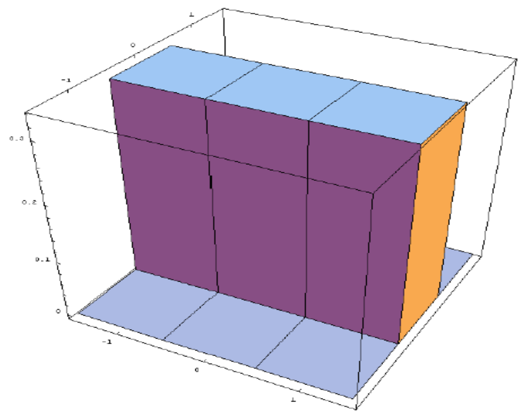
# Quasiprobabilities for finite quantum systems

Finite-dimensional quantum systems typically use a discrete phase space

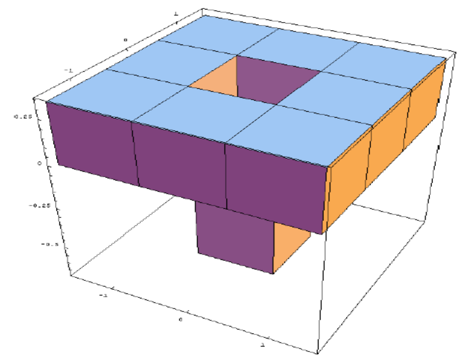



Gibbons, Hoffman, Wootters, PRA (2004); Gross, JMP (2006)

# Negativity and nonclassicality



Classical



Quantum

# Operationalizing nonclassicality

Negativity in a quasiprobability can be related to notions of nonclassicality

- Nonnegativity = simulatability

  Monte Carlo on the hidden variables

  Veitch, Ferrie, Gross, Emerson, NJP (2012)
  Mari and Eisert, PRL (2012)

- Negativity = contextuality

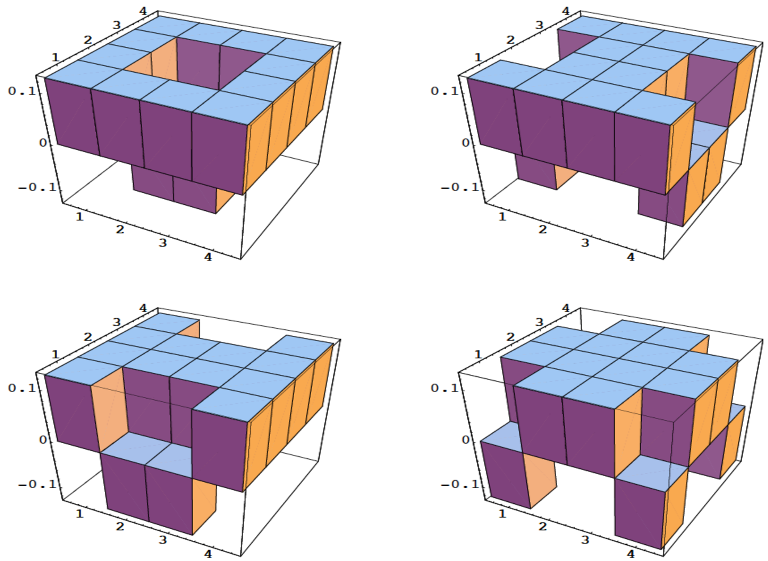
  Negativity in all quasiprobability representations is equivalent to a proof of contextuality

  Spekkens, PRL (2007)

- Negativity = magic

  Negative states are those that can be distilled to magic states, that can supplement Clifford gates to allow universal quantum computation

  Veitch, Mousavian, Gottesman, Emerson, NJP (2014)
  Howard, Wallman, Veitch, Emerson, Nature (2014)



## Structure of our result

› Can we push the boundary on simulatability?

1. Quantify negativity – review
2. Poly-precision estimators for Born rule probabilities
3. Born rule probabilities as quasiprobabilistic sum over trajectories
4. Construct a true probability distribution of trajectories as a Markov chain
5. Construct an unbiased estimator
6. Bound convergence of this estimator in terms of the amount of negativity

**Main Result**

Estimator converges to true quantum mechanical probability at a rate determined by the amount of negativity in the circuit

If the negativity is polynomially bounded -> efficiently yields a poly-precision estimate

Pashayan, Wallman, Bartlett (2015)

# Quantifying negativity

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# Quantifying negativity

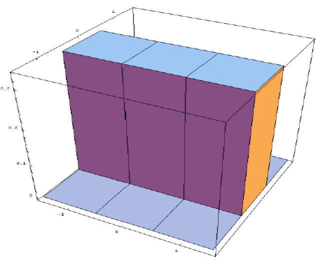
Veitch, Mousavian, Gottesman, Emerson, NJP (2014)

Define the *negativity* of a state: the 1-norm of its quasiprobability representation

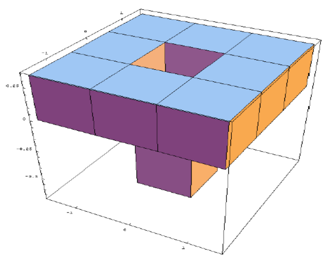
$$\mathcal{M}_\rho = \sum_{\lambda \in \Lambda} |W_\rho(\lambda)|$$

Negativity is multiplicitive, not additive (could take the log of this quantity)

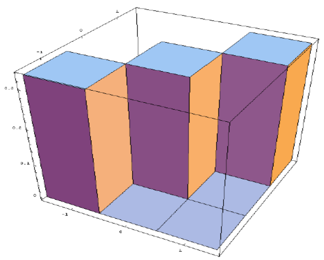
If $W_\rho$ is nonnegative, then $\mathcal{M}_\rho = 1$



$\mathcal{M}_\rho = 1$



$\mathcal{M}_\rho > 1$



$\mathcal{M}_\rho = 1$

# Negativity for states, unitaries, measurements

**Quantum States**

$$\mathcal{M}_\rho = \sum_{\lambda \in \Lambda} |W_\rho(\lambda)|$$

**Measurements (POVM elements)**

$$\mathcal{M}_E = \sum_{\lambda \in \Lambda} |W(E|\lambda)|$$

**Unitaries**

**Point negativity:** $$\mathcal{M}_U(\lambda) = \sum_{\lambda' \in \Lambda} |W_U(\lambda'|\lambda)|$$

**Negativity:** $$\mathcal{M}_U = \max_{\lambda \in \Lambda} \mathcal{M}_U(\lambda)$$

# Estimating measurement probabilities

# Trajectories in phase space

What do quasiprobabilities tell us about the probabilities of measurement outcomes?

$$p = \sum_{\lambda_0, \lambda_1, \ldots, \lambda_L} W(E|\lambda_L) W_{U_L}(\lambda_L|\lambda_{L-1}) \cdots W_{U_1}(\lambda_1|\lambda_0) W_\rho(\lambda_0)$$

Trajectories through phase space

Quasiprobability associated to each trajectory

If these were all nonnegative, it provides a natural estimation algorithm

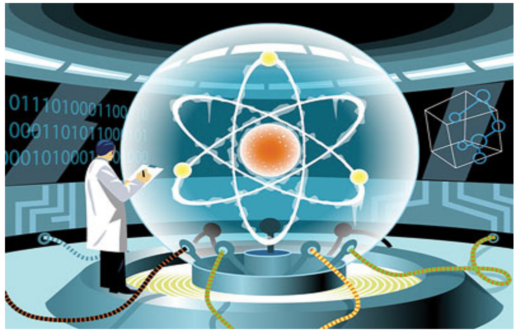
Veitch, Ferrie, Gross, Emerson, NJP (2012)
Mari and Eisert, PRL (2012)

But what if they are negative?

Can we estimate *p* by sampling from some true probability distribution?

# What's a good estimator?

What would make a good estimator of a probability associated with a measurement outcome?



Poly-precision estimator: for any fixed confidence, yields an estimate within $\varepsilon$ of the true Born rule probability using resources that scale polynomially in $1/\varepsilon$.

# True probabilities from quasiprobabilities

Quasiprobability for a trajectory

$$W(\vec{\lambda}) = W(E|\lambda_L)W_{U_L}(\lambda_L|\lambda_{L-1})\cdots W_{U_1}(\lambda_1|\lambda_0)W_\rho(\lambda_0)$$

May be negative, so how do we sample?

First attempt: sample from $\Pr(\vec{\lambda}) = \frac{|W(\vec{\lambda})|}{\mathcal{M}_c}$ $\qquad \mathcal{M}_c = \sum_{\vec{\lambda}} |W(\vec{\lambda})|$

Estimate of the probability for each trajectory is $\hat{q}_1 = \mathcal{M}_c \mathrm{Sign}[W(\vec{\lambda})]$

This gives an unbiased estimator, minimizes the range, and has the smallest variance of all estimators over the space of trajectories…

But is impossible to sample from!

# Our algorithm

Circuit with an efficient description (product input + output, local unitaries)

1. Sample initial point in trajectory from modified distribution

$$\Pr(\lambda_0) = |W_\rho(\lambda_0)|/\mathcal{M}_\rho$$

2. At each timestep $l$=$0$,…,$L$, sample from conditional distribution

$$\Pr(\lambda_l|\lambda_{l-1}) = |W_{U_l}(\lambda_l|\lambda_{l-1})|/\mathcal{M}_{U_l}(\lambda_{l-1})$$

3. Estimate based on single trajectory

$$\hat{p}_1(\lambda) = \mathcal{M}_\rho \mathrm{Sign}[W_\rho(\lambda_0)] \prod_{l=1}^{L} \left[\mathcal{M}_{U_l}(\lambda_{l-1}) \mathrm{Sign}[W_{U_l}(\lambda_l|\lambda_{l-1})]\right] W_E(\lambda_L)$$

# Properties of this estimate

Properties of estimator $\hat{p}_1(\lambda) = \mathcal{M}_\rho \mathrm{Sign}[W_\rho(\lambda_0)] \prod_{l=1}^{L} [\mathcal{M}_{U_l}(\lambda_{l-1}) \mathrm{Sign}[W_{U_l}(\lambda_l|\lambda_{l-1})]] W_E(\lambda_L)$

- Efficiently computable
- Unbiased estimator of Born rule probability

$$\begin{aligned}
\langle \hat{p}_1(\vec{\lambda}) \rangle &= \sum_{\vec{\lambda}} \hat{p}_1(\vec{\lambda}) \Pr(\vec{\lambda}) \\
&= \sum_{\vec{\lambda}} \hat{p}_1(\lambda) \frac{|W_\rho(\lambda_0)|}{\mathcal{M}_\rho} \prod_{l=1}^{L} \frac{|W_{U_l}(\lambda_l|\lambda_{l-1})|}{\mathcal{M}_{U_l}(\lambda_{l-1})} \\
&= \sum_{\vec{\lambda}} W_\rho(\lambda_0) \prod_{l=1}^{L} W_{U_l}(\lambda_l|\lambda_{l-1}) W_E(\lambda_L) \\
&= \Pr(E|\rho, U)
\end{aligned}$$

**Total negativity bound:**

$$\mathcal{M} = \mathcal{M}_\rho \prod_{l=1}^{L} \mathcal{M}_{U_l} \mathrm{max}_{\lambda_L} |W_E(\lambda_L)|$$

- Not a probability! Lies in the interval $[-\mathcal{M}, +\mathcal{M}]$

## Sampling and convergence

Compute $\hat{p}_1(\lambda)$ for *s* independent trajectories, take the average

- Unbiased, and bound to the interval $[-\mathcal{M}, +\mathcal{M}]$
- Use Hoeffding inequality for upper bound on convergence:

Average of *s* samples will be within $\epsilon$ of the quantum probability with probability $1 - \delta$ if the total number of samples taken is

$$s(\epsilon, \delta) = \frac{2}{\epsilon^2}\mathcal{M}^2 \ln(2/\delta)$$

If the total negativity grows at most polynomially in *N*, we have an efficient estimate of the quantum probability to within $\epsilon = 1/poly(N)$ , with an exponentially small failure probability

# Example

- Random 100-qutrit Clifford circuit
- Initialize with *k* magic states

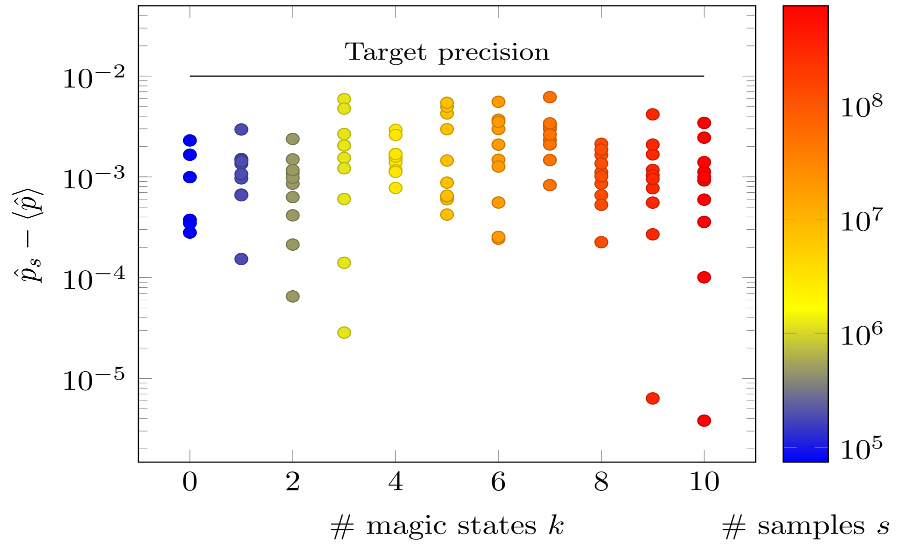
$$\frac{1}{\sqrt{3}}(|0\rangle + \xi|1\rangle + \xi^8|2\rangle)$$

$$\xi = \exp(2\pi i/9)$$

- Measure "0" on the first qutrit
- Number of samples chosen using

$$s(k) = \frac{2}{\epsilon^2} c^{2k} \ln(2/\delta)$$

- Target precision ε=0.01 with 95% confidence



# Examples of quasiprobabilities

- Odd-d qudit discrete Wigner function
  - All stabilizer states and measurements (+ some more) are nonnegative

Gibbons, Hoffman, Wootters, PRA (2004); Gross, JMP (2006)

- Real-valued qubit discrete Wigner function
  - Real stabilizer states and CSS-preserving unitaries

Delfosse, Guerin, Bian, Raussendorf, PRX (2015)
See also Bravyi, Smith, and Smolin, arXiv:1506.01396

- Qubit quasiprobabilities with nonnegative bases
  - 1, 2, 3, or 4 nonnegative bases and finite subgroups of SU(2) – no entanglement

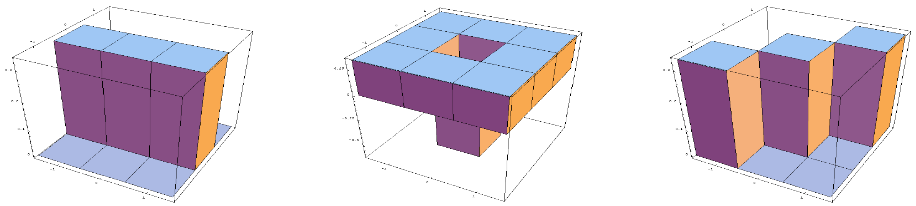
Wallman and Bartlett, PRA (2012)

- Continuous-variable Wigner function
  - Coherent states and squeezed states, linear optics and squeezing
  - Implications for BosonSampling?

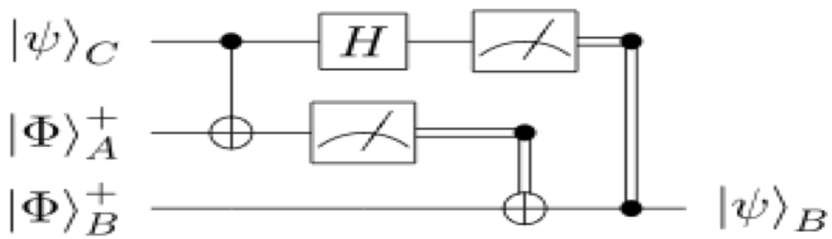
- Dual frames can be overcomplete – additional flexibility

# Conclusions and future directions

- Operational meaning of negativity: a measure that bounds the efficiency of a classical estimation of probabilities



- Efficient estimation vs *sparcity*
- Conditioning on intermediate measurements?
  - Naïve inclusion: calculating conditional probabilities requires exponential precision
  - Or make the conditional operation coherent, and delay measurement to the end: can add negativity
- From estimation to simulation




- An ontology for quantum computing?