Fast-forward approach to adiabatic quantum spin dynamics

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Samarkand, September 16, 2019

Part 1

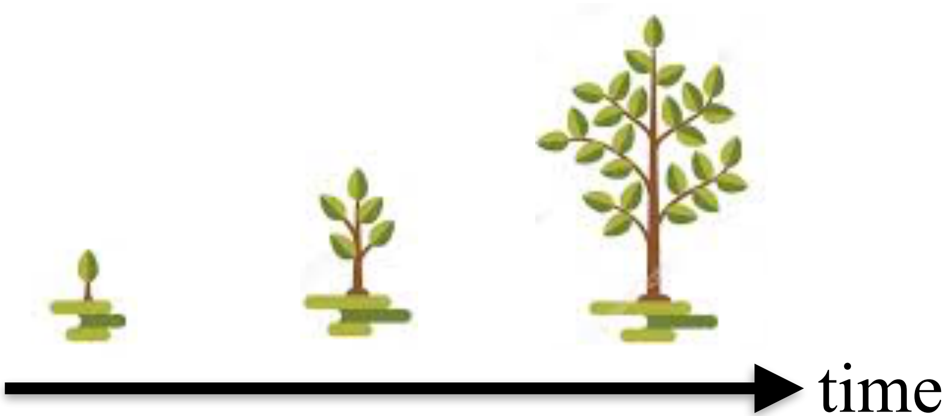
Fast forward of adiabatic spin dynamics: Landau-Zener transition and generation of entangled states

Part 2

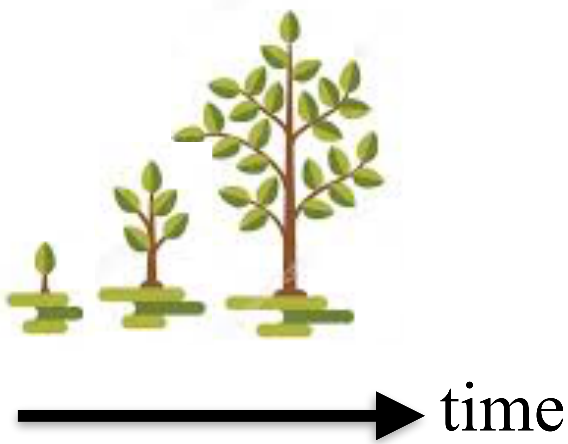
Fast forward of adiabatic quantum dynamics of spin clusters: Geometry-dependent driving interactions

What is the fast forward?

This terminology means to reproduce a series of events or a history of matters in a shortened time scale, like a rapid projection of movie films on the screen.



Reference evolution
(standard time scale)



Fast-forwarded evolution
(shortened time scale)

Fast forward (FF) can acquire reality with use of suitable protocols, e.g., by applying a well-designed external force, electromagnetic field, etc.

Theory and experiment of real fast forward (FF) is required wherever interesting dynamics is forced to be very slow, such as adiabatic quantum computation, adiabatic quantum annealing, etc.

FF scheme reproduces the adiabatic dynamics in a shortened time scale, leaving neither residual oscillation nor disturbance.

Theory of FF of adiabatic dynamics is proposed by Masuda and Nakamura (2010~) and includes as a special limit the recent theory of shortcut-to-adiabaticity (STA) by Rice (2003, 2005), Berry (2009), Muga, et al (2010~), Jarzynski (2013), *et al*.

References for the general scheme of fast forward:

[1] S. Masuda and K. Nakamura, Phys. Rev. A **78**, 062108 (2008).

[2] S. Masuda and K. Nakamura, Proc. R. Soc. A **466**, 1135 (2010).

[3] S. Masuda and K. Nakamura, Phys. Rev. A **84**, 043434 (2011).

References for application of the fast forward scheme:

[1] A. Khujakulov and K. Nakamura,

Phys. Rev. A **93**, 022101 (2016).

[2] K. Nakamura, A. Khujakulov, S. Avazbaev, and S. Masuda,
Phys. Rev. A **95**, 062108 (2017).
[3] S. Masuda1, K. Nakamura, and M. Nakahara,

New J. Phys. **20,** 025008 (2018).

[4] G. Babajanova, J. Matrasulov, and K. Nakamura,

Phys. Rev. E **97**, 042104 (2018).

In the present lectures, consider **spin systems** characterized by

a **slowly time-changing parameter** $R(t)$

such as the magnetic field, exchange interaction, etc.

References related to the present lectures:

[1] I. Setiawan, Bobby Eka Gunara, S. Masuda and K. Nakamura, Phys.Rev. A 96, 052106 (2017).

[2] I. Setiawan, Bobby Eka Gunara, A. Avazbaev, and K. Nakamura, Phys.Rev. A 99, 062116 (2019).

Let's forcibly consider the eigenvalue problem for the time-independent Schroedinger equation(TDSE)

$$H_0(R)\begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix} = E(R)\begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix}$$

$$R(t) = R_0 + \epsilon t \qquad \epsilon \ll 1$$

(adiabatically-changing parameter)

Quasi-adiabatic wave function

$$\Psi_0(R(t)) = \begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix} e^{-\frac{i}{\hbar}\int_0^t E(R(t'))dt'} e^{i\xi(t)},$$

$$\begin{aligned} \xi(t) &= i\int_0^t dt' \Big(C_1^* \frac{\partial C_1}{\partial t} + ... + C_N^* \frac{\partial C_N}{\partial t} \Big) \\ &= i\epsilon \int_0^t dt' \Big(C_1^* \frac{\partial C_1}{\partial R} + ... + C_N^* \frac{\partial C_N}{\partial R} \Big). \end{aligned}$$

(adiabatic phase)

To make the quasi-adiabatic wave function satisfy TDSE, the regularization of the Hamiltonian is necessary:

$$H_0^{reg}(R(t)) = H_0(R(t)) + \epsilon\tilde{\mathcal{H}}_n(R(t)).$$

Then TDSE becomes

$$i\hbar\frac{\partial}{\partial t}\Psi_0(R(t)) = (H_0 + \epsilon\tilde{\mathcal{H}}_n)\Psi_0(R(t)). \quad (2.5)$$

Here $\tilde{\mathcal{H}}_n$ is the n-th state-dependent regularization term

Let's expand Eq.(2.5) with respect to ϵ

In order of $O(1)$

$$H_0\Psi_0 = E\Psi_0, \tag{2.6}$$

and in order of $O(\epsilon^1)$

$$\tilde{\mathcal{H}}_n \begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix} = i\hbar \begin{pmatrix} \frac{\partial C_1(R)}{\partial R} \\ \vdots \\ \frac{\partial C_N(R)}{\partial R} \end{pmatrix} - i\hbar \left(\sum_{j=1}^{N} C_j^* \frac{\partial C_j}{\partial R} \right) \begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix} \tag{2.7}$$

This is **our core equation** to determine the regularization terms $\tilde{\mathcal{H}}_n$

Quasi-adiabatic function and TDSE
under the regularized Hamiltonian are characterized by a slow time scale. However, they work well not only for short time but also for any long time.

Our practical interest lies in their long-time (T~O(1/\epsilon)) behaviors where
the adiabatic parameter value $R(t)$ shows a recognizable change.

Therefore we consider <u>fast forwarding</u> with use of a large time-scaling factor:

The fast forward state is defined by

$$\Psi_{FF}(t) = \begin{pmatrix} C_1(R(\Lambda(t))) \\ \vdots \\ C_N(R(\Lambda(t))) \end{pmatrix} e^{-\frac{i}{\hbar}\int_0^t E((R(\Lambda(t'))))dt'} e^{i\xi((R(\Lambda(t))))} \tag{2.8}$$

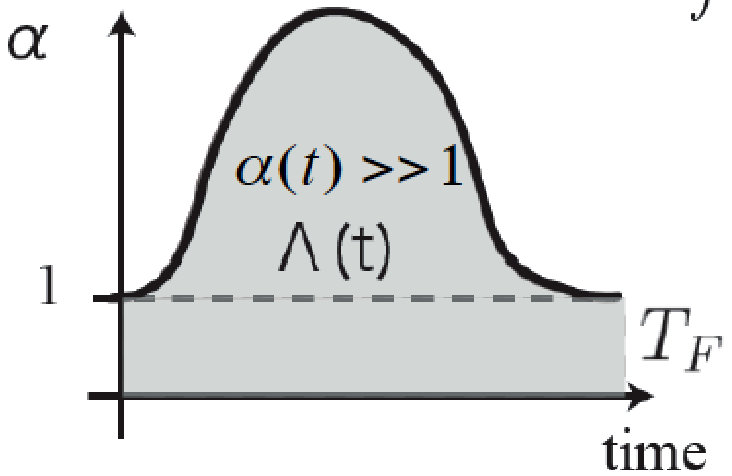
where $\Lambda(t)$ is an advanced time defined by

$$\Lambda(t) = \int_0^t \alpha(t')dt', \tag{2.9}$$

$\Lambda(t) = t$ in case of no scaling

The explicit expression for $\alpha(t)$ in the fast-forward range $(0 \leq t \leq T_{FF})$ is typically given by [2] as :

$$\alpha(t) = \bar{\alpha} - (\bar{\alpha} - 1)\cos\left(\frac{2\pi}{T_{FF}}t\right), \qquad (2.11)$$



$$T = \int_0^{T_{FF}} \alpha(t)dt. \quad (\alpha >> 1)$$

T(~*O*(1/\epsilon)): a very long time for each event (**adiabatic** spin inversion, LZ transition, **adiabatic** transition from product to entangled states, etc) to be completed.

We can show that FF state satisfies TDSE under the FF Hamiltonian:

$$i\hbar\frac{\partial\Psi_{FF}}{\partial t} = \Big(H_0(R(\Lambda(t))) + v(t)\tilde{\mathcal{H}}_n(R(\Lambda(t)))\Big)\,\Psi_{FF} \equiv H_{FF}\Psi_{FF}. \qquad (2.13)$$

Detailed proof will be given in Part 2.

$$\begin{aligned} v(t) &= \lim_{\epsilon\to 0,\alpha\to\infty} \epsilon\alpha(t) \\ &= \bar{v}\left(1-\cos\frac{2\pi}{T_{FF}}t\right), \end{aligned} \tag{2.14}$$

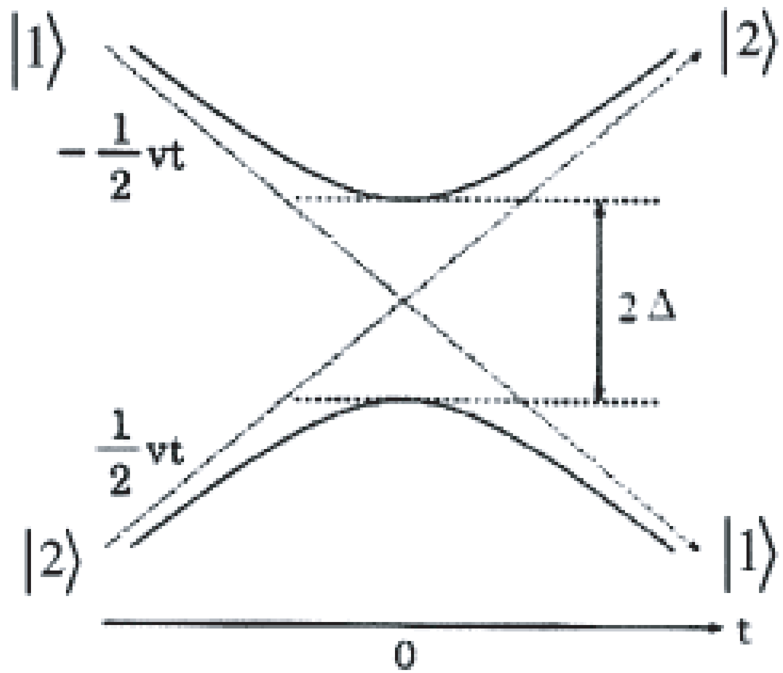
where $\bar{v} = \lim_{\epsilon\to 0,\alpha\to\infty} \epsilon\bar{\alpha}(= finite)$ is the mean of $v(t)$.

$$\begin{aligned} R(\Lambda(t)) &= R_0 + \lim_{\epsilon\to 0,\bar{\alpha}\to\infty} \varepsilon\Lambda(t) \\ &= R_0 + \int_0^t v(t')dt' \\ &= R_0 + \bar{v}\left[t - \frac{T_{FF}}{2\pi}\sin\left(\frac{2\pi}{T_{FF}}t\right)\right], \\ &\quad \text{for } 0 \le t \le T_{FF}. \end{aligned}$$

single-spin dynamics: Landau-Zener model

$$H_0(R(t)) = \frac{1}{2}\boldsymbol{\sigma}\cdot\mathbf{B} = \frac{1}{2}\begin{pmatrix} R(t) & \Delta \\ \Delta & -R(t) \end{pmatrix}$$

$$R(t) = R_0 + \epsilon t$$



$$P_{LZ} = \exp(-\frac{\pi\Delta^2}{2v})$$

$$\Psi_0^{\pm} = \begin{pmatrix} C_1^{\pm} \\ C_2^{\pm} \end{pmatrix} = \begin{pmatrix} -\Delta/s_{\pm} \\ \frac{R \mp \sqrt{R^2+\Delta^2}}{s_{\pm}} \end{pmatrix}, \tag{2.22}$$

where

$$s_{\pm} \equiv \left[2\sqrt{R^2 + \Delta^2} \left(\sqrt{R^2 + \Delta^2} \mp R \right) \right]^{1/2}. \tag{2.23}$$

Now we choose one of the states with λ_+ and Ψ_0^+, and consider the adiabatic dynamics where $R = R_0 + \epsilon t$. The adiabatically evolving state is :

$$\Psi_0(t) = \begin{pmatrix} -\frac{\Delta}{s_+} \\ \frac{R-\sqrt{R^2+\Delta^2}}{s_+} \end{pmatrix} e^{-\frac{i}{\hbar}\int_0^t \frac{\sqrt{R^2+\Delta^2}}{2} dt'} e^{\xi(t)}. \tag{2.24}$$

$$\tilde{\mathcal{H}}_n \begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix} = i\hbar \begin{pmatrix} \frac{\partial C_1(R)}{\partial R} \\ \vdots \\ \frac{\partial C_N(R)}{\partial R} \end{pmatrix} - i\hbar \left(\sum_{j=1}^{N} C_j^* \frac{\partial C_j}{\partial R} \right) \begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix}.$$

Noting that $\tilde{\mathcal{H}}_{ij}$ is traceless ($\tilde{\mathcal{H}}_{11} =$ - $\tilde{\mathcal{H}}_{22}$) and Hermitian ($\tilde{\mathcal{H}}_{21}^* = \tilde{\mathcal{H}}_{12}$), Eq.(2.7) constitutes a rank = 2 linear algebraic equation for two unknowns ($\tilde{\mathcal{H}}_{11}$ and $\tilde{\mathcal{H}}_{12}$). With

$$\frac{\partial C_1}{\partial R} = -\frac{1}{2\sqrt{2}} \frac{\Delta}{Q^{5/2}} (Q - R)^{\frac{1}{2}}$$

$$\frac{\partial C_2}{\partial R} = \frac{1}{2\sqrt{2}} \frac{(Q - R)^{\frac{1}{2}} (Q + R)}{Q^{5/2}},$$

$$Q = \sqrt{R^2 + \Delta^2}$$

driving Hamiltonian:

$$\mathcal{H} = v(t)\tilde{\mathcal{H}} = \begin{pmatrix} 0 & v(t)i\frac{\hbar}{2}\frac{\Delta}{Q^2} \\ -v(t)i\frac{\hbar}{2}\frac{\Delta}{Q^2} & 0 \end{pmatrix} \tag{2.28}$$

fast-forward Hamiltonian:

$$H_{FF} = \begin{pmatrix} \frac{R(\Lambda(t))}{2} & \frac{\Delta}{2} + v(t)i\frac{\hbar}{2}\frac{\Delta}{Q^2} \\ \frac{\Delta}{2} - v(t)i\frac{\hbar}{2}\frac{\Delta}{Q^2} & -\frac{R(\Lambda(t))}{2} \end{pmatrix}. \tag{2.29}$$

The fast forward state is obtained from Eq.(2.8) as

$$\Psi_{FF} = \begin{pmatrix} C_1^+(\Lambda(t)) \\ C_2^+(\Lambda(t)) \end{pmatrix} e^{-\frac{i}{\hbar}\int_0^t \frac{\sqrt{R(\Lambda(t'))^2+\Delta^2}}{2} dt'}. \tag{2.30}$$

The total driving magnetic field is written as

$$\mathbf{B}_{FF}(t) = \begin{pmatrix} \Delta \\ -v(t)\hbar\frac{\Delta}{R(\Lambda(t))^2+\Delta^2} \\ R(\Lambda(t)) \end{pmatrix}. \tag{2.31}$$

Fast forward of adiabatic LZ state-change with no transition!

Coupled two-spin systems

Candidate regularization Hamiltonian (9 possible parameters)

$$\tilde{\mathcal{H}} = \tilde{J}_1\sigma_1^x\sigma_2^x + \tilde{J}_2\sigma_1^y\sigma_2^y + \tilde{J}_3\sigma_1^z\sigma_2^z + \tilde{W}_1\big(\sigma_1^x\sigma_2^y + \sigma_1^y\sigma_2^x\big)$$

$$+ \tilde{W}_2\big(\sigma_1^y\sigma_2^z + \sigma_1^z\sigma_2^y\big) + \tilde{W}_3\big(\sigma_1^z\sigma_2^x + \sigma_1^x\sigma_2^z\big) + \tfrac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\tilde{\mathbf{B}},$$

With use of the bases, $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$
its matrix form becomes

$$\tilde{\mathcal{H}} = \begin{pmatrix} \tilde{J}_3 + \tilde{B}_z & \frac{1}{2}(\tilde{B}_x - i\tilde{B}_y) - i\tilde{W}_2 + \tilde{W}_3 & \frac{1}{2}(\tilde{B}_x - i\tilde{B}_y) - i\tilde{W}_2 + \tilde{W}_3 & \tilde{J}_1 - \tilde{J}_2 - i2\tilde{W}_1 \\ \frac{1}{2}(\tilde{B}_x + i\tilde{B}_y) + i\tilde{W}_2 + \tilde{W}_3 & -\tilde{J}_3 & \tilde{J}_1 + \tilde{J}_2 & \frac{1}{2}(\tilde{B}_x - i\tilde{B}_y) + i\tilde{W}_2 - \tilde{W}_3 \\ \frac{1}{2}(\tilde{B}_x + i\tilde{B}_y) + i\tilde{W}_2 + \tilde{W}_3 & \tilde{J}_1 + \tilde{J}_2 & -\tilde{J}_3 & \frac{1}{2}(\tilde{B}_x - i\tilde{B}_y) + i\tilde{W}_2 - \tilde{W}_3 \\ \tilde{J}_1 - \tilde{J}_2 + i2\tilde{W}_1 & \frac{1}{2}(\tilde{B}_x + i\tilde{B}_y) - i\tilde{W}_2 - \tilde{W}_3 & \frac{1}{2}(\tilde{B}_x + i\tilde{B}_y) - i\tilde{W}_2 - \tilde{W}_3 & \tilde{J}_3 - \tilde{B}_z \end{pmatrix}.$$

The explicit expression for the regularization Hamiltonian greatly reduces the number of unknowns and helps us to solve our core equation.

(A) Simple transverse Ising model

Reference Hamiltonian

$$H_0 = J(R(t))\sigma_1^z\sigma_2^z - \frac{1}{2}(\sigma_1^x + \sigma_2^x)B_x(R(t)) \qquad (3.3)$$

By using this bases : $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$, we have

$$H_0 = \begin{pmatrix} J & -\frac{B_x}{2} & -\frac{B_x}{2} & 0 \\ -\frac{B_x}{2} & -J & 0 & -\frac{B_x}{2} \\ -\frac{B_x}{2} & 0 & -J & -\frac{B_x}{2} \\ 0 & -\frac{B_x}{2} & -\frac{B_x}{2} & J \end{pmatrix} \qquad (3.4)$$

where the eigenvalue : $-J$, J, $-\sqrt{J^2+B_x^2}$, and $\sqrt{J^2+B_x^2}$). The normalized eigenvector are respectively:

$$\begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} \frac{B_x}{2\sqrt{B_x^2+J^2+J\sqrt{B_x^2+J^2}}} \\ \frac{\sqrt{B_x^2+J^2}+J}{2\sqrt{B_x^2+J^2+J\sqrt{B_x^2+J^2}}} \\ \frac{\sqrt{B_x^2+J^2}+J}{2\sqrt{B_x^2+J^2+J\sqrt{B_x^2+J^2}}} \\ \frac{B_x}{2\sqrt{B_x^2+J^2+J\sqrt{B_x^2+J^2}}} \end{pmatrix}, \quad \text{and}$$

ground state

$$\begin{pmatrix} \frac{B_x}{2\sqrt{B_x^2+J^2-J\sqrt{B_x^2+J^2}}} \\ \frac{-\sqrt{B_x^2+J^2}+J}{2\sqrt{B_x^2+J^2-J\sqrt{B_x^2+J^2}}} \\ \frac{-\sqrt{B_x^2+J^2}+J}{2\sqrt{B_x^2+J^2-J\sqrt{B_x^2+J^2}}} \\ \frac{B_x}{2\sqrt{B_x^2+J^2-J\sqrt{B_x^2+J^2}}} \end{pmatrix}.$$

The ground state changes its nature from the product state (c1=c2=c3=c4=1/2) to entangled state(c2=c3=1/\sqrt(2), c1=c4=0), as *Bx* tends 0 and *J* increases from 0.

Due to the symmetry $C_1 = C_4$ and $C_2 = C_3$, Eq.(2.7) for the regularization terms reduces to

$$i\hbar \frac{\partial C_4}{\partial R} = \tilde{\mathcal{A}}_1 C_4 + \tilde{\mathcal{A}}_2 C_2, \tag{3.6}$$
$$i\hbar \frac{\partial C_2}{\partial R} = \tilde{\mathcal{A}}_3 C_4 + \tilde{\mathcal{A}}_4 C_2,$$

where $\tilde{\mathcal{A}}_1 = (\tilde{\mathcal{H}}_{11} + \tilde{\mathcal{H}}_{14})$, $\tilde{\mathcal{A}}_2 = (\tilde{\mathcal{H}}_{12} + \tilde{\mathcal{H}}_{13})$, $\tilde{\mathcal{A}}_3 = (\tilde{\mathcal{H}}_{21} + \tilde{\mathcal{H}}_{24})$, and $\tilde{\mathcal{A}}_4 = (\tilde{\mathcal{H}}_{22} + \tilde{\mathcal{H}}_{23})$.

To solve two-component simultaneous liner equations, there are several possibilities of solutions. We pick up the cases when the equation for unknown $\{\tilde{J}, \tilde{W}, \tilde{B}\}$ is regular.

$$\tilde{J}_3 = \frac{aC_4 + bC_2}{C_4^2 - C_2^2} = 0$$

$$\tilde{W}_2 = \frac{i(aC_2 + bC_4)}{2\left(C_2^2 - C_4^2\right)},$$

To be explicit,

$$\tilde{W}_2 = \frac{-J\,\frac{\partial B_x}{\partial R} + B_x\,\frac{\partial J}{\partial R}}{4\left(B_x^2 + J^2\right)}.$$

Driving and and fast-forward Hamiltonians

$$\mathcal{H} = \begin{pmatrix} 0 & -iv(t)\tilde{W}_2 & -iv(t)\tilde{W}_2 & 0 \\ iv(t)\tilde{W}_2 & 0 & 0 & iv(t)\tilde{W}_2 \\ iv(t)\tilde{W}_2 & 0 & 0 & iv(t)\tilde{W}_2 \\ 0 & -iv(t)\tilde{W}_2 & -iv(t)\tilde{W}_2 & 0 \end{pmatrix}, \tag{3.11}$$

$$\begin{aligned} H_{FF} &= J(R(\Lambda(t)))\sigma_1^z\sigma_2^z - \frac{1}{2}(\sigma_1^x + \sigma_2^x)B_x(R(\Lambda(t))) \\ &+ v(t)\tilde{W}_2(R(\Lambda(t)))(\sigma_1^y\sigma_2^z + \sigma_1^z\sigma_2^y). \end{aligned} \tag{3.12}$$

$$
\begin{aligned}
R(\Lambda(t)) &= R_0 + \lim_{\epsilon \to 0, \bar{\alpha} \to \infty} \varepsilon \Lambda(t) \\
&= R_0 + \int_0^t v(t')dt' \\
&= R_0 + \bar{v}\left[t - \frac{T_{FF}}{2\pi}\sin\left(\frac{2\pi}{T_{FF}}t\right)\right], \\
&\quad \text{for } 0 \le t \le T_{FF}.
\end{aligned}
$$

The adiabatic parameter R changes by the value $\bar{v}\, T_{\mathrm{FF}}$ ($\sim O(R_0)$) in an arbitrary short time T_{FF}.

$$c_1=c_2=c_3=c_4=1/2 \longrightarrow c_2=c_3=1/\sqrt{2},\ c_1=c_4=0$$

In this example, FF dynamics generates the entangled (Bell) state from the initial product state, quickly, and not adiabatically, leaving neither residual oscillations nor disturbances.

(B) Model for generation of entangled state

$$H_0 = J\sigma_1^z\sigma_2^z + \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\mathbf{B}, \qquad (3.27)$$

which can generate an entangled state from the product state. In Eq.(3.27) $\mathbf{B} = (B_x, B_y, B_z)$ with $B_z = B_z(R(t))$. B_x, B_y and J are assumed constants. Arranging the bases as $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$, we obtain

$$H_0 = \begin{pmatrix} J+B_z & \frac{B_x}{2} - i\frac{B_y}{2} & \frac{B_x}{2} - i\frac{B_y}{2} & 0 \\ \frac{B_x}{2} + i\frac{B_y}{2} & -J & 0 & \frac{B_x}{2} - i\frac{B_y}{2} \\ \frac{B_x}{2} + i\frac{B_y}{2} & 0 & -J & \frac{B_x}{2} - i\frac{B_y}{2} \\ 0 & \frac{B_x}{2} + i\frac{B_y}{2} & \frac{B_x}{2} + i\frac{B_y}{2} & J-B_z \end{pmatrix}.$$

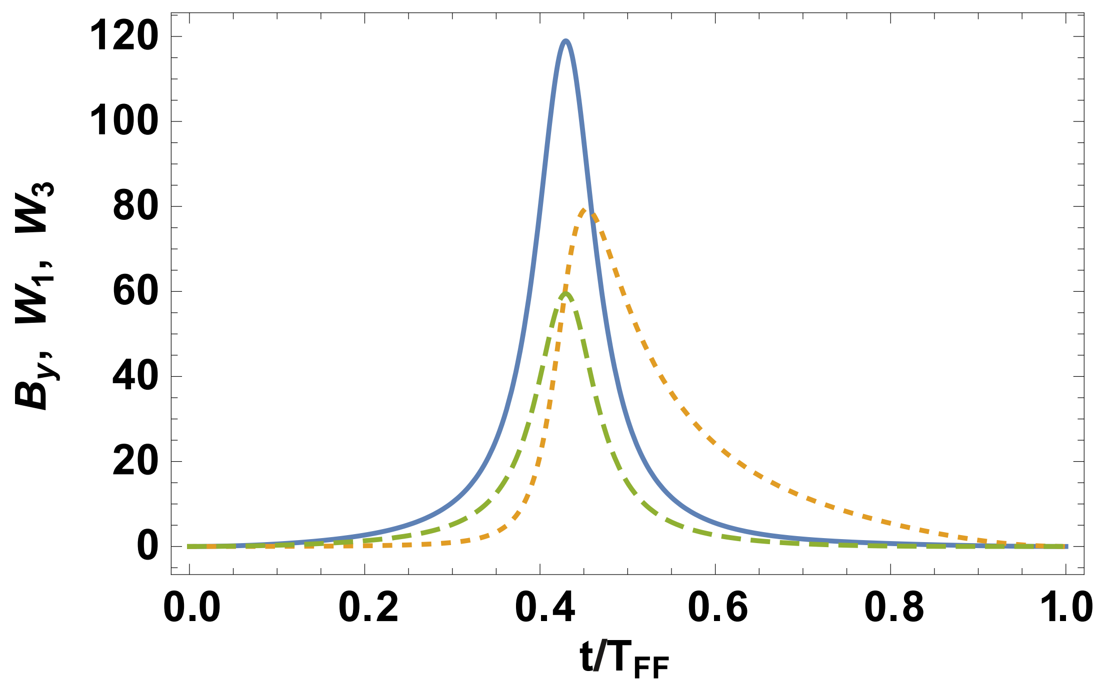
As *Bz* decreases from zero to some negative value, the ground state changes from the product state (c4=1, c1=c2=c3=0) to the entangled one (c1=c4=0, c2=c3=1/\sqrt(2)).

Result:

Fast-forward Hamiltonian

$$
\begin{aligned}
H_{\mathrm{FF}} = & -J\sigma_1^z\sigma_2^z - \tfrac{1}{2}\left(\sigma_1^z+\sigma_2^z\right)B_z \\
& - \tfrac{1}{2}\left(\sigma_1^x+\sigma_2^x\right)B_x(R(\Lambda(t))) \\
& + v(t)\tilde{W}_2(R(\Lambda(t)))\left(\sigma_1^y\sigma_2^z+\sigma_1^z\sigma_2^y\right) \\
& + \tfrac{1}{2}\left(\sigma_1^y+\sigma_2^y\right)v(t)\tilde{B}_y(R(\Lambda(t))),
\end{aligned}
$$

Driving interaction



FF dynamics generates the entangled (Bell) state from the initial product state, quickly and not adiabatically.

Wavefunction solution of TDSE exactly agrees with the time dependence of the eigenstate.

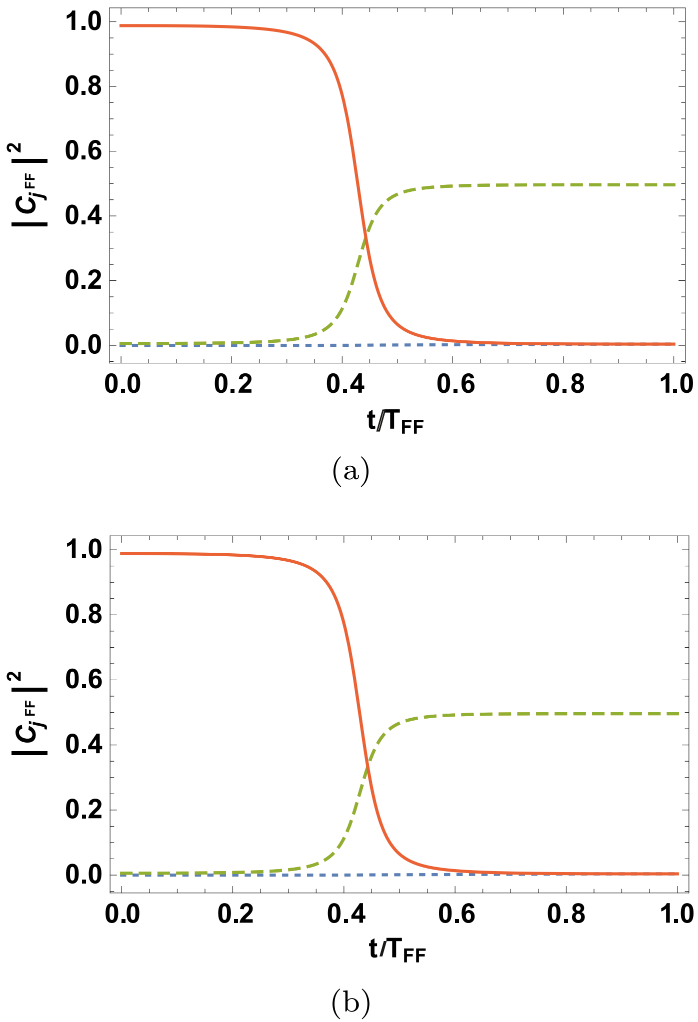


FIG. 4: The time dependence of $|C_4^{FF}|^2$ (solid line), $|C_2^{FF}|^2$ (dashed line), $|C_3^{FF}|^2$ (dashed line), and $|C_1^{FF}|^2$ (dotted line):(a) Obtained by solving TDSE ; (b) Obtained from eigenvectors.

Relationship between *<u>H</u>n* and Demirplak-Rice-Berry's state-independent counter-diabatic(CD) term *H*

If there is an *n*-independent regularization term among $\{\underline{H}n\}$, we define $H=v(t)H(R(L(t)))$ with use of $v(t)= \frac{\partial R(\Lambda(t))}{\partial t}$
Then our core equation becomes

$$\mathcal{H}\Psi_0 = i\hbar\frac{\partial}{\partial t}\Psi_0 - i\hbar\sum_{j=1}^{N} C_j^* \frac{\partial C_j}{\partial t}\Psi_0,$$

which can be rewritten as

$$\mathcal{H}|n\rangle = i\hbar\frac{\partial}{\partial t}|n\rangle - i\hbar|n\rangle\langle n|\frac{\partial}{\partial t}|n\rangle,$$

$$\mathcal{H}\sum_n |n\rangle\langle n| = i\hbar\sum_n \frac{\partial}{\partial t}|n\rangle\langle n| - i\hbar\sum_n |n\rangle\langle n|\frac{\partial}{\partial t}|n\rangle\langle n|. \tag{2.18}$$

Noting the completeness condition for the eigenstates : $\sum_n |n\rangle\langle n| = 1$, we have

$$\mathcal{H} = i\hbar\sum_n \left(\frac{\partial}{\partial t}|n\rangle\langle n| - |n\rangle\langle n|\frac{\partial}{\partial t}|n\rangle\langle n|\right), \tag{2.19}$$

Our regularization Hamiltonian, multiplied with velocity function, corresponds to Demirplak-Rice-Berry's counter-diabatic (CD) term.