Tunable Quantum Networks

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OUTLINE
Part I: Linear waves in networks

- Quantum networks in physics
- Evolution equations on network
- Quantum graphs: Nonrelativistic case
- Quantum graphs: Relativistic case
- PT-symmetric quantum graphs
- Tunable wave propagation in quantum networks

Network science

Networks are used for modelling broad variety of complex systems:
From macromolecules to WWW, social, economic, political and ecological systems.

Networks can be modelled in terms of metric graphs.

Graph is characterized by its topology, a connection rule for graph bonds.

Network science

How networks are attacked?

Statistical physics based approach
Statistical distributions of bonds and vertices and their dependence on graphs topology

Discrete, or tight binding approach:
Tight binding Hamiltonian on metric graphs

Continuum approach:
Evolution equations on metric graphs

What is quantum Network?

No standard definition of quantum network.

Depends on the topic where network appears

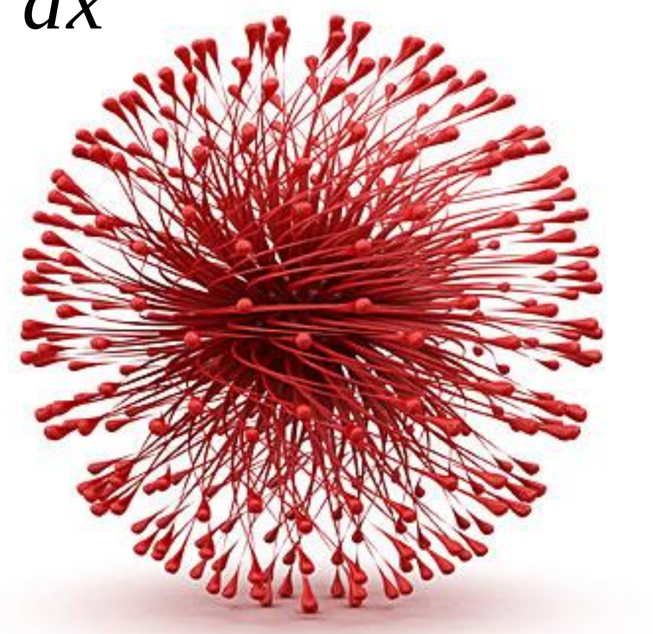
Our definition:

Any branched structure (network) where the particles/waves/phenomena are described in terms of quantum mechanical wave equations

Quantum Networks in Optics: Microwave Networks

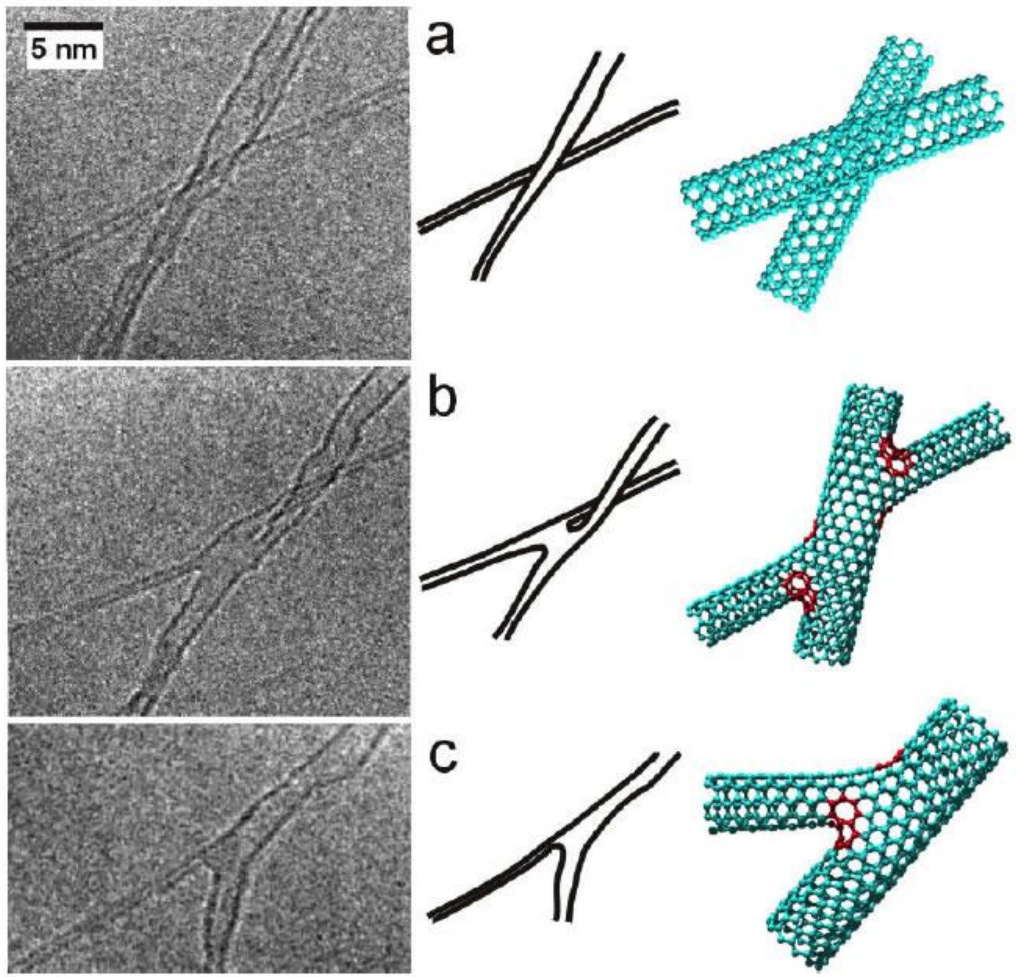
Wave transport in optical fibers is described by Helmholtz equation:

$$-\frac{d^2}{dx^2}\Psi = k^2\Psi$$



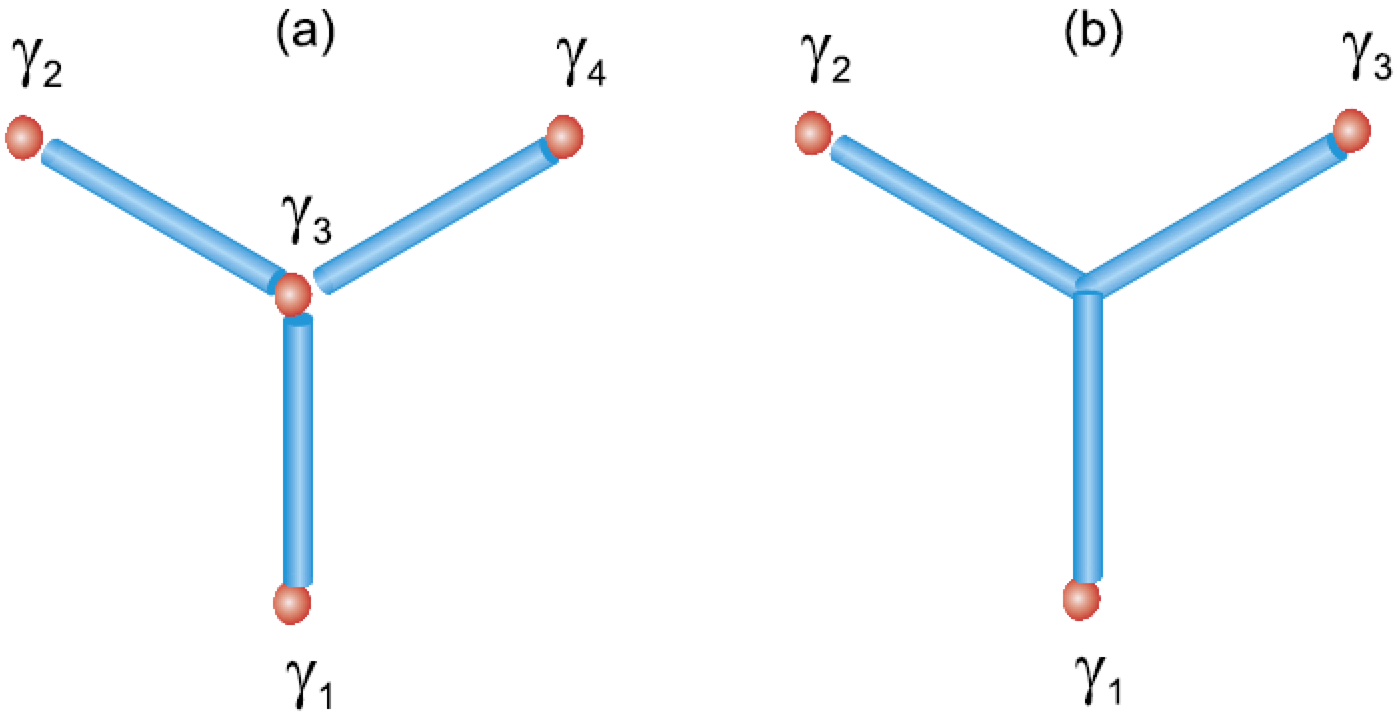
O . Hul *et al Phys. Rev.* E **69** 056205 (2004)

Quantum networks in condensed matter: Branched carbon nanotube

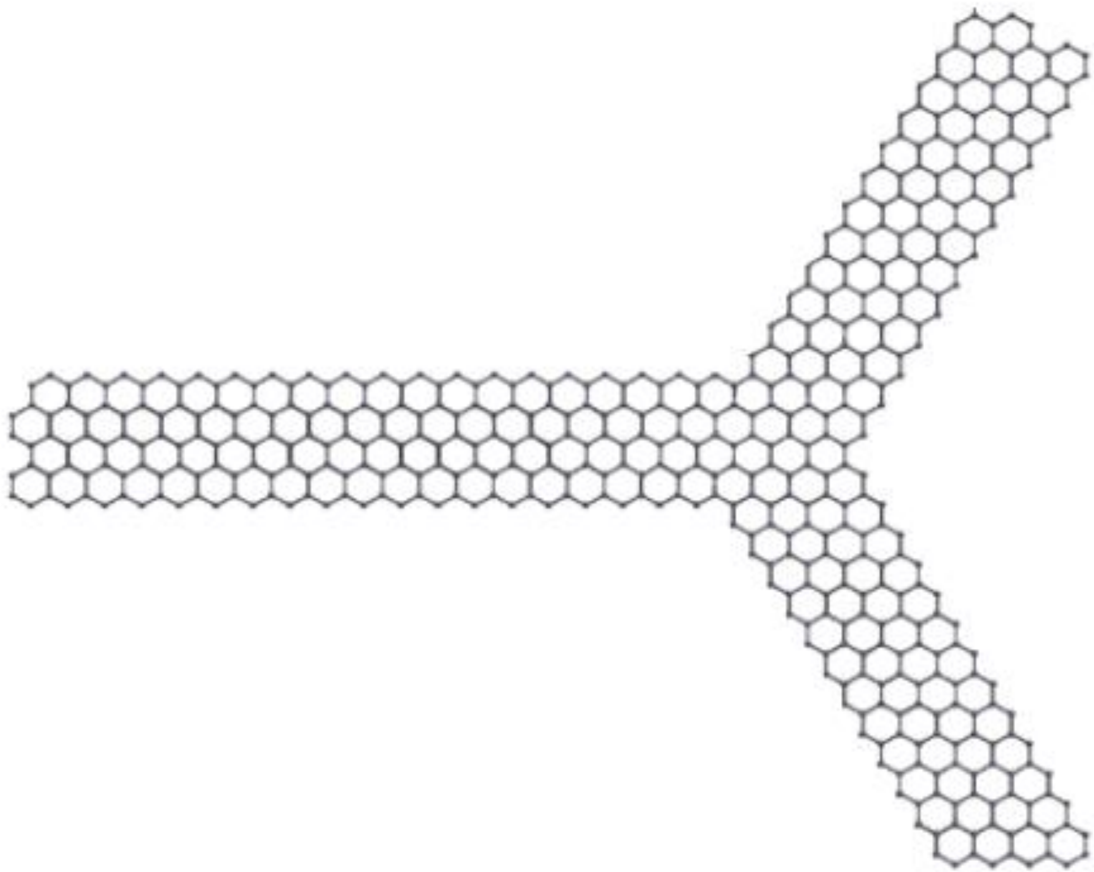


M. Terrones, F. Banhart, N. Grobert, J. C. Charlier, H. Terrones and P. M. Ajayan, Physical Review Letters 89, 75505, 2002.

Quantum networks in condensed matter: Majorana wire networks



Quantum networks in condensed matter: Branched graphene nanoribbon



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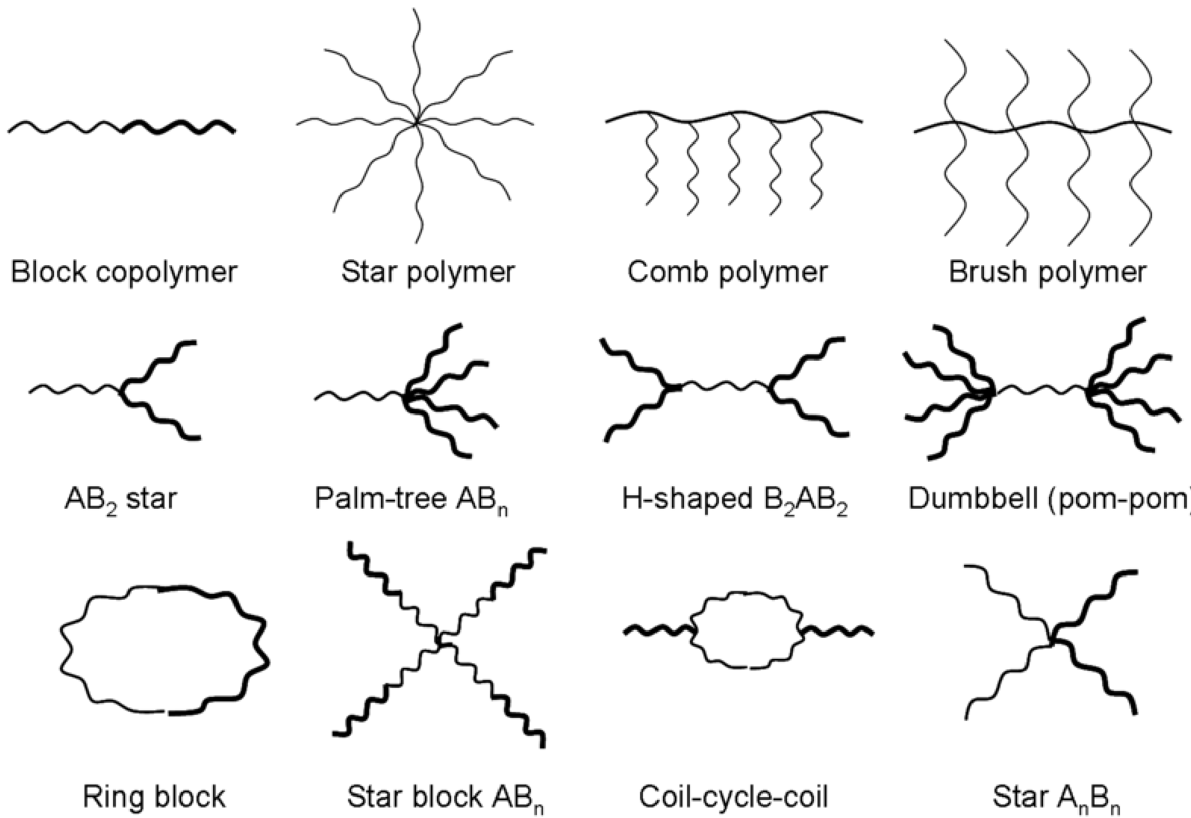
Quantum networks in polymers: Exciton dynamics in conducting polymers



Quantum networks in quantum information

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PAPER

A quantum network stack and protocols for reliable entanglement-based networks

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Keywords: quantum networks, quantum communication, quantum internet

Abstract

We present a stack model for breaking down the complexity of entanglement-based quantum networks. More specifically, we focus on the structures and architectures of quantum networks and not on concrete physical implementations of network elements. We construct the quantum network stack in a hierarchical manner comprising several layers, similar to the classical network stack, and identify quantum networking devices operating on each of these layers. The layers responsibilities range from establishing point-to-point connectivity, over intra-network graph state generation, to inter-network routing of entanglement. In addition we propose several protocols operating on these layers. In particular, we extend the existing intra-network protocols for generating arbitrary graph states to ensure reliability inside a quantum network, where here reliability refers to the capability to compensate for devices failures. Furthermore, we propose a routing protocol for quantum routers which enables the generation of arbitrary graph states across network boundaries. This protocol, in correspondence with classical routing protocols, can compensate dynamically for failures of routers, or even complete networks, by simply re-routing the given entanglement over alternative paths. We also consider how to connect quantum routers in a hierarchical manner to reduce complexity, as well as reliability issues arising in connecting these quantum networking devices.

Quantum networks in quantum information

NATURE|Vol 453|19 June 2008|doi:10.1038/nature07127 INSIGHT REVIEW

The quantum internet

H. J. Kimble[1]

Quantum networks provide opportunities and challenges across a range of intellectual and technical frontiers, including quantum computation, communication and metrology. The realization of quantum networks composed of many nodes and channels requires new scientific capabilities for generating and characterizing quantum coherence and entanglement. Fundamental to this endeavour are quantum interconnects, which convert quantum states from one physical system to those of another in a reversible manner. Such quantum connectivity in networks can be achieved by the optical interactions of single photons and atoms, allowing the distribution of entanglement across the network and the teleportation of quantum states between nodes.

In the past two decades, a broad range of fundamental discoveries have been made in the field of quantum information science, from a quantum algorithm that places public-key cryptography at risk to a protocol for the teleportation of quantum states[1]. This union of quantum mechanics and information science has allowed great advances in the understanding of the quantum world and in the ability to control coherently individual quantum systems[2]. Unique ways in which quantum systems process and distribute information have been identified, and powerful new perspectives for understanding the complexity and subtleties of quantum dynamical phenomena have emerged.

In the broad context of quantum information science, quantum networks have an important role, both for the formal analysis and the physical implementation of quantum computing, communication and metrology[2–5]. A notional quantum network based on proposals in refs 4, 6 is shown in Fig. 1a. Quantum information is generated, processed and stored locally in quantum nodes. These nodes are linked by quantum channels, which transport quantum states from site to site with high fidelity and distribute entanglement across the entire network. As an extension of this idea, a 'quantum internet' can be envisaged; with only moderate processing capabilities, such an internet could accomplish tasks that are impossible in the realm of classical physics, including the distribution of 'quantum software'[7].

Apart from the advantages that might be gained from a particular algorithm, there is an important advantage in using quantum connectivity, as opposed to classical connectivity, between nodes. A network of quantum nodes that is linked by classical channels and comprises k nodes each with n quantum bits (qubits) has a state space of dimension $k2^n$, whereas a fully quantum network has an exponentially larger state space, 2^{kn}. Quantum connectivity also provides a potentially powerful means to overcome size-scaling and error-correlation problems that would limit the size of machines for quantum processing[8]. At any stage in the development of quantum technologies, there will be a largest size attainable for the state space of individual quantum processing units, and it will be possible to surpass this size by linking such units together into a fully quantum network.

A different perspective of a quantum network is to view the nodes as components of a physical system that interact by way of the quantum channels. In this case, the underlying physical processes used for quantum network protocols are adapted to simulate the evolution of quantum many-body systems[9]. For example, atoms that are localized at separate nodes can have effective spin–spin interactions catalysed by single-photon pulses that travel along the channels between the nodes[10]. This 'quantum wiring' of the network allows a wide range for the effective hamiltonian and for the topology of the resultant 'lattice'. Moreover, from this perspective, the extension of entanglement across quantum networks can be related to the classical problem of percolation[11].

These exciting opportunities provide the motivation to examine research related to the physical processes for translating the abstract illustration in Fig. 1a into reality. Such considerations are timely because scientific capabilities are now passing the threshold from a learning phase with individual systems and advancing into a domain of rudimentary functionality for quantum nodes connected by quantum channels.

In this review, I convey some basic principles for the physical implementation of quantum networks, with the aim of stimulating the involvement of a larger community in this endeavour, including in systems-level studies. I focus on current efforts to harness optical processes at the level of single photons and atoms for the transportation of quantum states reliably across complex quantum networks.

Two important research areas are strong coupling of single photons and atoms in the setting of cavity quantum electrodynamics (QED)[12] and quantum information processing with atomic ensembles[13], for which crucial elements are long-lived quantum memories provided by the atomic system and efficient, quantum interfaces between light and matter. Many other physical systems are also being investigated and are discussed elsewhere (ref. 2 and websites for the Quantum Computation Roadmap (http://qist.lanl.gov/qcomp_map.shtml), the SCALA Integrated Project (http://www.scala-ip.org/public) and Qubit Applications (http://www.qubitapplications.com)).

A quantum interface between light and matter

The main scientific challenge in the quest to distribute quantum states across a quantum network is to attain coherent control over the interactions of light and matter at the single-photon level. In contrast to atoms and electrons, which have relatively large long-range interactions for their spin and charge degrees of freedom, individual photons typically have interaction cross-sections that are orders of magnitude too small for non-trivial dynamics when coupled to single degrees of freedom for a material system.

The optical physics community began to address this issue in the 1990s, with the development of theoretical protocols for the coherent transfer of quantum states between atoms and photons in the setting of cavity QED[2,14,15]. Other important advances have been made in the past

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Quantum networks in quantum information

Quantum Networks: From Quantum Cryptography to Quantum Architecture

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ABSTRACT
As classical information technology approaches limits of size and functionality, practitioners are searching for new paradigms for the distribution and processing of information. Our goal in this Introduction is to provide a broad view of the beginning of a new era in information technology, an era of quantum information, where previously underutilized quantum effects, such as quantum superposition and entanglement, are employed as resources for information encoding and processing. The ability to distribute these new resources and connect distant quantum systems will be critical. We present an overview of network implications for quantum communication applications, and for quantum computing. This overview is a selection of several illustrative examples, to serve as motivation for the network research community to bring its expertise to the development of quantum information technologies.

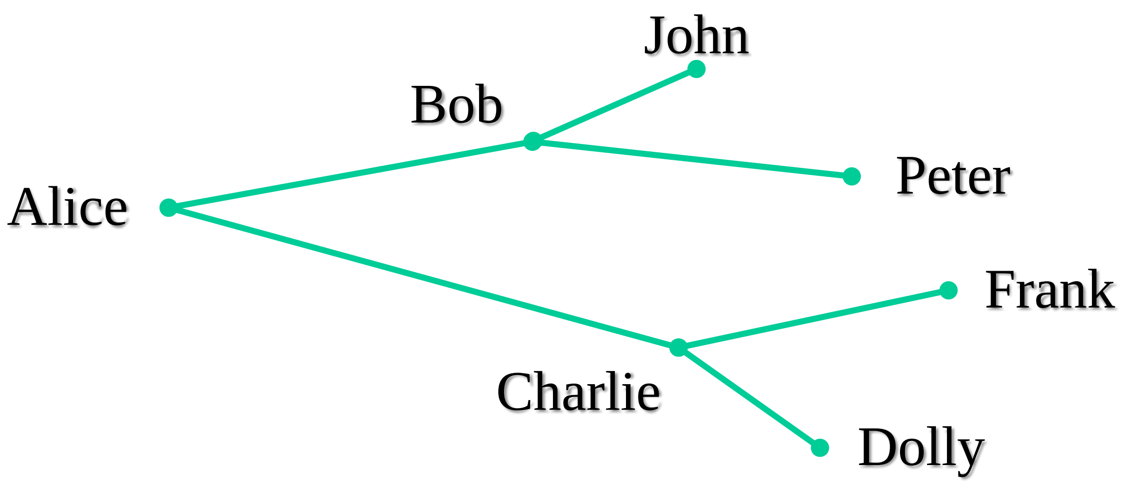
1. INTRODUCTION
The past century has been influenced tremendously by both quantum mechanics and information technology. Quantum effects are pervasive in technology, central to functioning of many ubiquitous devices today, such as lasers, integrated circuits, fluorescent lights, and magnetic resonance imaging (MRI) machines, to name a few. Following the development of ARPANET under DARPA's sponsorship in the '60s, we have witnessed an explosive growth of information technology. Integrated circuits and the internet have changed our lives. If the last century was the era of quantum mechanics and information technology, the 21[st] century will be an era of quantum information, where previously underutilized quantum effects, such as quantum superposition and entanglement, will be essential resources for information encoding and processing. The ability to distribute these new resources and connect distant quantum systems will be critical. In this paper we present a brief overview of network implications for quantum communication applications, such as cryptography, and for quantum computing. This overview is not meant to be exhaustive. Rather, it is a selection of several illustrative examples, to serve as motivation for the network research community to bring its expertise to the development of quantum information technologies.

2. QUANTUM COMMUNICATION
2.1 Quantum Cryptography
Novel quantum systems that manipulate, store and transmit information based on laws of quantum mechanics are now being explored in many physical systems at the level of individual atoms, photons, and electrons. Quantum effects such as entanglement and superposition of quantum states, the no-cloning theorem, non-locality principles, etc. are exploited in quantum cryptography, quantum communication, and quantum computation. The most mature of all quantum information technologies, quantum cryptography [21], takes advantage of the no-cloning property [30] of quantum states to implement unbreakable secure cryptosystems.

To ensure that communications are secure, two parties exchanging sensitive information over an insecure communication channel must use a cryptographic protocol. The sending party has to use a cryptographic key to encode the information (encryption) and the receiving party has to use a key to decode the information (decryption). If a third party acquires the decoding key, he/she will be able to decode the information. There are two distinct ways to distribute the keys – *private* and *public*. In *private*, or *symmetrical*, key cryptosystems the parties have to share a secret key before they send and receive a message. If the key is the same length as the message, randomly generated every time, and is used only once, that cryptographic algorithm is called the *one-time pad* or *Vernam cipher*, the only existing provably secure cryptosystem at this time. Modern *public* or *asymmetrical* key cryptographic systems are based on two matching keys – a public

Quantum networks in quantum information: Quantum communication and cryptography via branched channels

Quantum network



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Linear waves in networks: Quantum graph concept

Particle/wave dynamics in networks can be described in terms of Schrödinger equation on graphs. In this case the latter called quantum graph.

Quantum graph concept

Quantum networks

↓

Quantum graphs

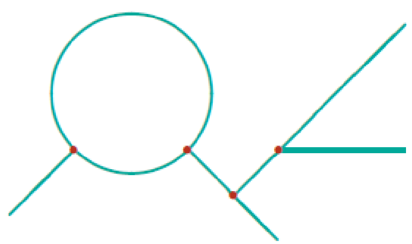
↓

Schrödinger equation on metric graph

Quantum graph concept

The idea of investigating quantum particles confined to a graph was first suggested by L. Pauling and worked out by Ruedenberg and Scherr in 1953 in a model of aromatic hydrocarbons

The concept extends, however, to graphs of *arbitrary shape*



Hamiltonian: $-\frac{\partial^2}{\partial x_j^2} + v(x_j)$
on graph edges,
boundary conditions at vertices

and what is important, it became *practically important after* experimentalists learned in the last two decades to fabricate tiny graph-like structure for which this is a good model

Metric graphs

A graph with the bonds which can be assigned length,

$$o < l_b :< D$$

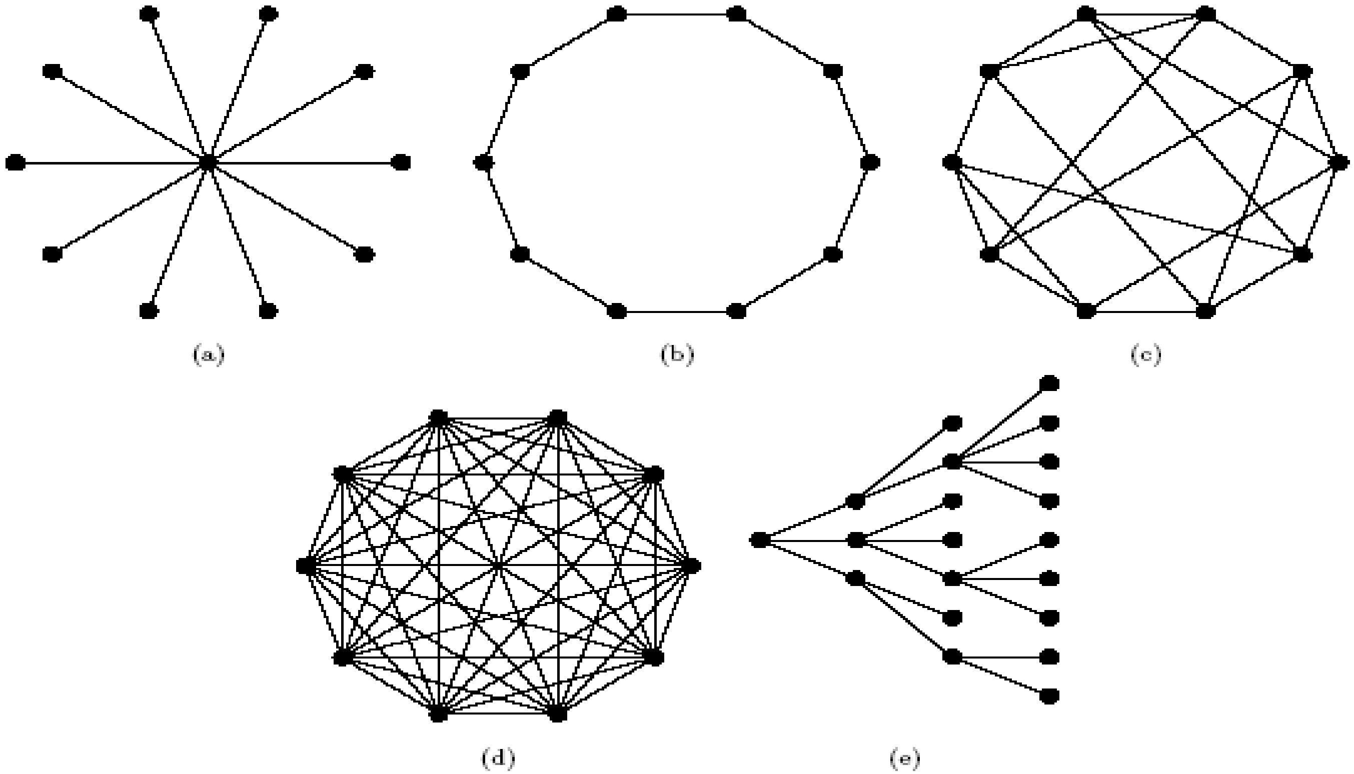
is called metric graph

Graphs and their topology

The topology of the graph, that is, the way the vertices and bonds are connected is given in terms of the $V \times V$ connectivity matrix $C_{i,j}$ (sometimes referred to as the adjacency matrix) which is defined as:

$$C_{i,j} = C_{j,i} = \left\{ \begin{array}{l} 1 \text{ if } i,j \text{ are connected} \\ 0 \text{ otherwise} \end{array} \right\}, \; i,j = 1,...,V.$$

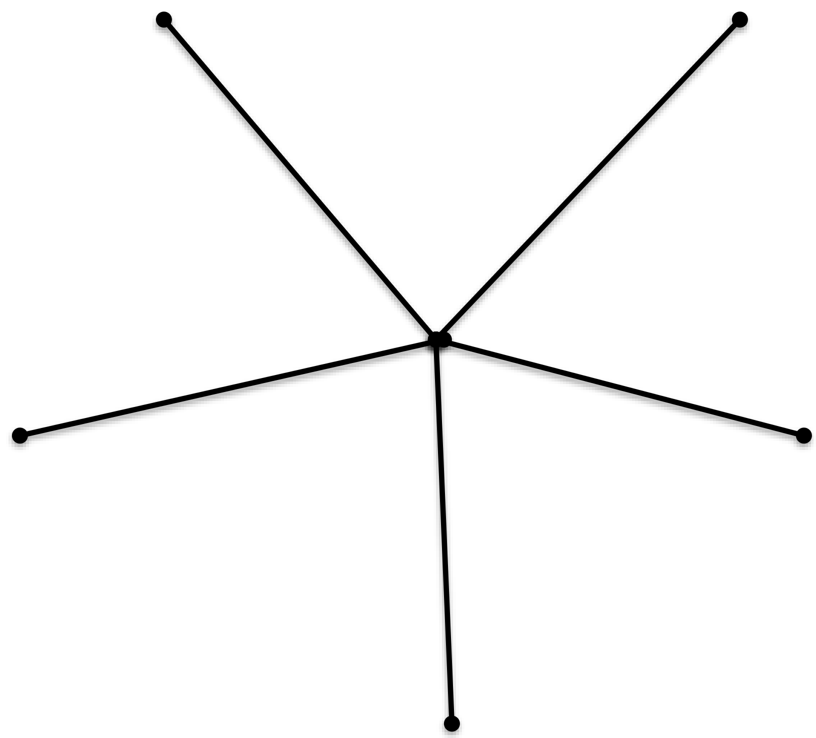
Graphs and their topology



(a) star graph (*B = 10, V = 11*),
(b) ring graph (*B = 10, V = 10*),
(c) *v*-regular graph with *v = 4* (*B = 20, V = 10*),
(d) complete (or well-connected) graph (*B = 45, V = 10*),
(e) tree graph (*B = 19, V = 20*).

Constructing quantum graphs from finite interval (wires)

Metric graph as a collection of interval glues to each other according to connectivity matrix



Constructing quantum graphs from finite interval (wires)

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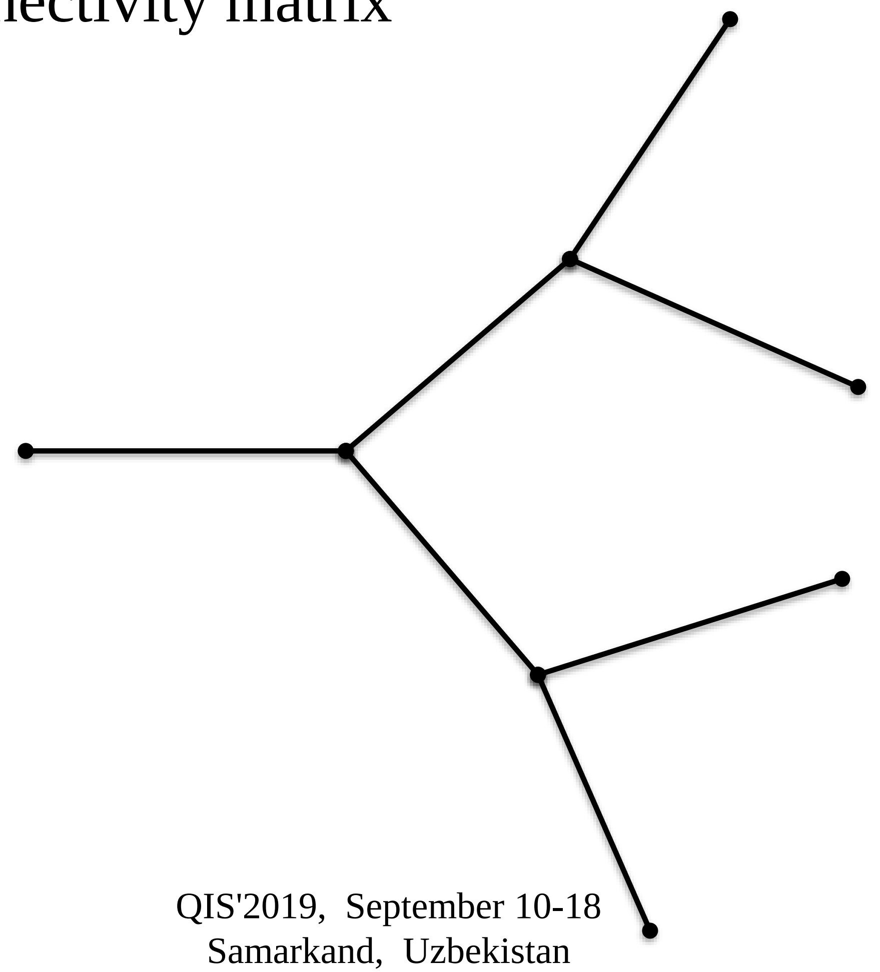


Evolution equation on graphs

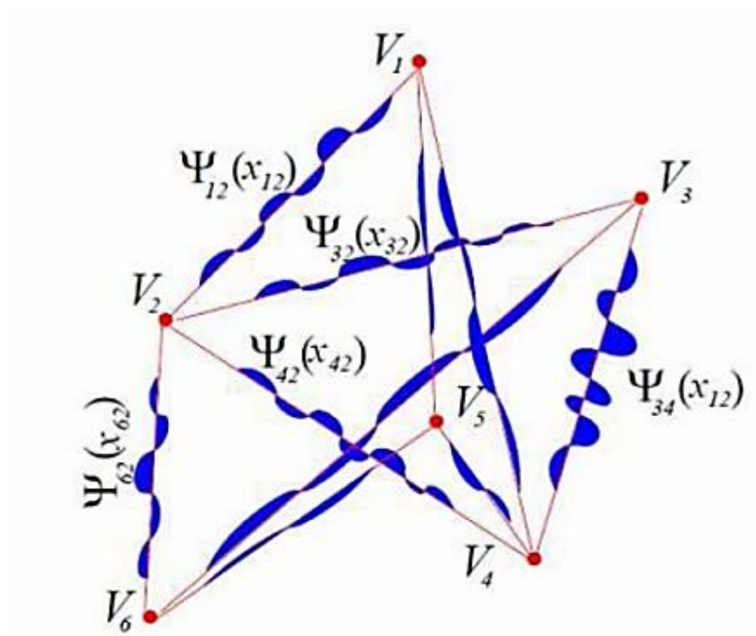
$$i\frac{\partial\psi}{\partial t} = H\psi$$

where H is the Shrödinger or Dirac operator

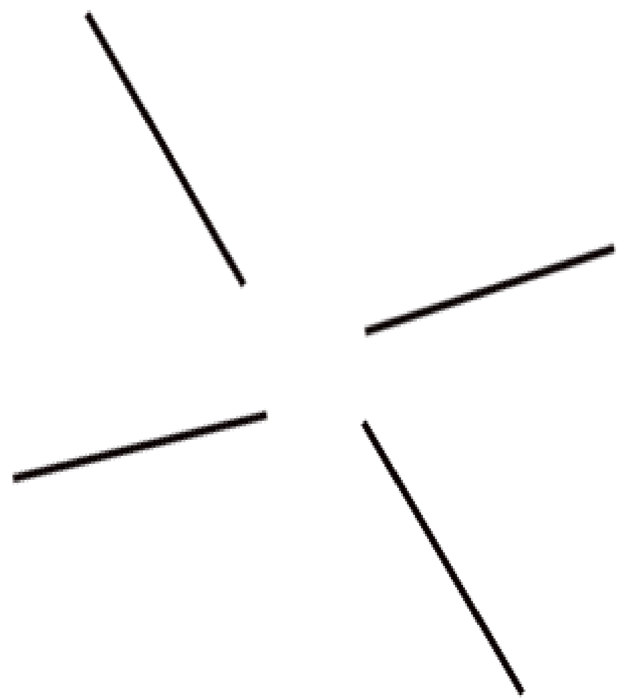
Wave equation on graphs: Wave function

Wave function Ψ is a B-component vector

$$\left(\Psi_{b_1}(x_{b_1}), \Psi_{b_2}(x_{b_2}), \ldots, \Psi_{b_B}(x_{b_B})\right)^T$$



Wave equation on graphs: Vertex Boundary conditions



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Differential operators on graphs

For given self-adjoint differential operator on graph D skew-Hermitian form can be constructed as

$$\Omega(\varphi,\phi)=\left\langle D\varphi,\phi\right\rangle-\left\langle\varphi,D\phi\right\rangle$$

V.Kostrykin, R.Schrader, J. Phys. A. **32** 595 (1999)

Boundary conditions

$$\mathbf{A}\psi(0)+\mathbf{B}\psi'(0)=0$$

where A and B are two n×n matrices

V.Kostrykin, R.Schrader, J. Phys. A: Math. Gen. **32** (1999) 595–630.

What have been studied in the context of quantum graphs so far?

- **Mathematical formulation of the problem, boundary conditions** Exner (1988), Kostrykin Schrader (1999), Seba (2000)
- **Quantum chaos in networks:**
- Kottos, Smilansky (1999), Gaspard (2004,) Gnutzmann (2006)
- **Inverse problems**
- Kurasov 2001, Smilansky (2004), Cheon (2010)
- **Casimir effect**
- Kaplan (2005), Matrasulov (2006), Bellazini (2007)
- **Quantum hall effect**
- Gaspasrd (2008)
- **Microwave networks (networks of optical fibers)**
- Hull (2007)

The Schrödinger equation on graphs: Wave function

For each bond $b = (i, j)$ a coordinate $x_{i,j}$ which indicates the position along the bond is assigned. The variable $x_{i,j}$ takes the value 0 at the vertex i and the value $L_{i,j} \equiv L_{j,i}$ at the vertex j while $x_{j,i}$ is zero at j and $L_{i,j}$ at i. We have thus defined the length matrix $L_{i,j}$ with matrix elements different from zero, whenever $C_{i,j} \neq 0$ and $L_{i,j} = L_{j,i}$ for $b = 1, ..., B$.

The wavefunction Ψ is a B−component vector and can be written as

$$\left(\Psi_{b_1}(x_{b_1}), \Psi_{b_2}(x_{b_2}), ..., \Psi_{b_B}(x_{b_B})\right)^T$$

where the set $\{b_i\}_{i=1}^{B}$ consists of B different bonds

The Schrödinger equation on graphs: Boundary Conditions

The wave function must satisfy boundary conditions at the vertices, which ensure continuity (uniqueness) and current conservation. For every $i = 1, \dots, V$:

- *Continuity*:

$$\Psi_{i,j}(x)\Big|_{x=0} = \varphi_i, \qquad \Psi_{i,j}(x)\Big|_{x=L_{i,j}} = \varphi_j$$

For all $i < j$ and $C_{i,j} \neq 0$

- *Current conservation*

$$-\sum_{j<i} C_{i,j} \left.\frac{d\Psi_{i,j}(x)}{dx}\right|_{x=L_{i,j}} + \sum_{j>i} C_{i,j} \left.\frac{d\Psi_{i,j}(x)}{dx}\right|_{x=0} = \lambda_i \varphi_i$$

The parameters λ_i are free parameters which determine the type of the boundary conditions.

The special case of zero λ_i's, corresponds to Neumann boundary conditions. Dirichlet boundary conditions are introduced when all the $\lambda_i = \infty$.

The Schrödinger equation on graphs: Solutions

At any bond *b = (i, j)* the component *b* can be written in terms of its values on the vertices *i* and *j* as

$$\Psi_{i,j} = \frac{1}{\sin kL_{i,j}}\left(\varphi_i \sin\left[k\left(L_{i,j} - x\right)\right] + \varphi_j \sin kx\right) C_{i,j}, \qquad i < j.$$

The current conservation condition leads to

$$-\sum_{j<i} \frac{kC_{i,j}}{\sin(kL_{i,j})}\left(-\varphi_j + \varphi_i \cos(kL_{i,j})\right) + \sum_{j>i} \frac{kC_{i,j}}{\sin(kL_{i,j})}\left(-\varphi_i \cos(kL_{i,j}) + \varphi_j\right) = \lambda_i \varphi_i, \qquad \forall i.$$

The Schrödinger equation on graphs: Eigenvalues

Spectral equation

$$\det\left(h_{i,j}(k)\right) = 0$$

where

$$h_{i,j} = \begin{cases} -\sum_{m\neq i} C_{i,m}\cot\left(kL_{i,m}\right) - \frac{\lambda_i}{k}, & i = j \\ C_{i,j}\left(\sin\left(kL_{i,j}\right)\right)^{-1}, & i \neq j. \end{cases}$$

Quantum star graph

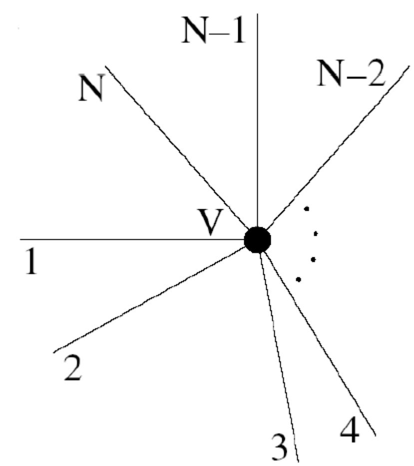
A graphs of the most simplest topology is so-called star-graph. It consist of three or more bonds connected at the single vertex which can be called central vertex. Other ones are called edge vertices. The eigenvalue problem for a star graph with N bonds is given by the following Schrödinger equation:

$$-i\frac{d^2}{dx^2}\phi_j(y) = k^2\phi_j(y), \qquad j = 1, \dots, N.$$

We assign for each bond j a coordinate y_j which indicates the position along the bond and takes the value 0 at the vertex V and the value l_j at the edge vertex.

The boundary conditions for the star graph are

$$\begin{cases} \phi_1|_{y=0} = \phi_2|_{y=0} = \dots = \phi_N|_{y=0}, \\ \phi_1|_{y=l_1} = \phi_2|_{y=l_2} = \dots = \phi_N|_{y=l_N} = 0, \\ \sum_{j=1}^{N} \frac{d}{dy}\phi_j\Big|_{y=0} = 0. \end{cases}$$



J.P.Keating, Contemp. Math., 415, 191 (2006)

Quantum star graph

The eigenvalues can be found by solving the following equation

$$\sum_{j=1}^{N} \cot(k l_j) = 0$$

where corresponding eigenfunctions are given as

$$\phi_j^{(n)}(y) = \frac{B_n}{\sin k_n l_j} \sin k_n (l_j - y)$$

with normalization coefficient

$$B_n = \sqrt{\frac{2}{\sum_j \frac{l_j - \sin 2k_n l_j}{\sin^2 k_n l_j}}}$$

Quantum transport

Probability current

$$J_k(x,t) = \frac{1}{2i}\left[\Psi_k^*(x,t)\,\frac{d\Psi_k}{dx} - \Psi_k(x,t)\,\frac{d\Psi_k^*}{dx}\right]$$

$$\Psi_k(x,t) = \sum_n e^{-iE_n t}\,\psi_k^{(n)}(x)$$

Quantum transport

Conductivity

$$\sigma_k(x) = \frac{1}{\omega}\int_0^{\infty} d\tau e^{-i\omega\tau}\langle[J_k(x,0), J_k(x,\tau)]\rangle$$

$$\langle[J_k(x,0), J_k(x,\tau)]\rangle$$

$$= \int_0^{L_k} dx[J_k(x,0)J_k(x,\tau) - J_k(x,0)J_k(x,\tau)]$$

PT-symmetric quantum mechanics

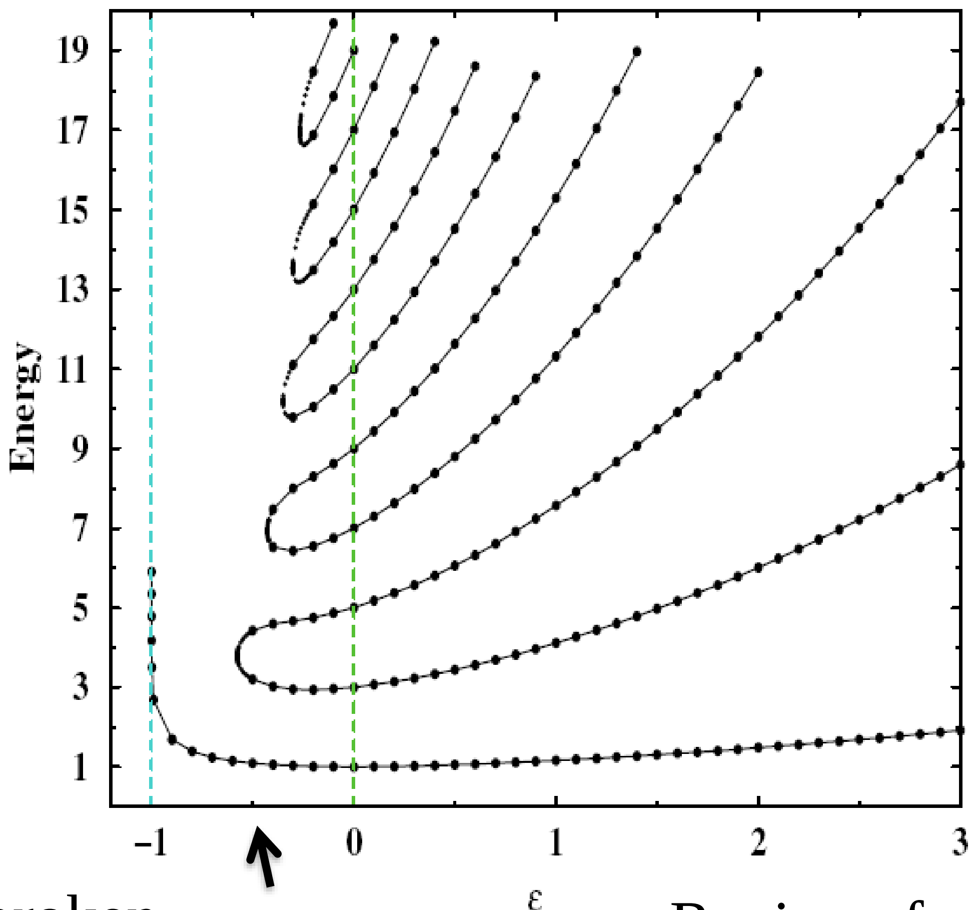
Since from the beginning of quantum physics people believed that to have real energy spectrum Hamiltonian operator should be Hermitian (self-adjoint). This fact was considered as necessary and enough condition for the realness of the spectrum. However such faith was broken in 1998 by Bender and Boettcher.

PT-symmetric quantum mechanics

In 1998, Bender and Boettcher [Phys. Rev. Lett. ***80*** *5243 (1998)] showed that quantum systems with a non-Hermitian Hamiltonian can have a set of eigenstates with real eigenvalues (a real spectrum).*

In other words, they found that the Hermiticity of the Hamiltonian is not a necessary condition for the realness of its eigenvalues, and new quantum mechanics can be constructed based on such Hamiltonians.

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$



Region of broken
PT symmetry

PT phase
transition

Region of unbroken
PT symmetry

C. Bender and S. Boettcher, *PRL* **80**, 5243 (1998)

What is PT-symmetry?

The PT-symmetry of the Hamiltonian means that it commutes with the time reversal operator *T* and the parity operator *P*:

$$\hat{P}\hat{T}\hat{H} = \hat{H}\hat{P}\hat{T}$$

$$\hat{P} + \hat{\mathbf{r}}\hat{P} = -\hat{\mathbf{r}}$$

$$\hat{P} + \hat{\mathbf{p}}\hat{P} = -\hat{\mathbf{p}}$$

$$\hat{P} + \hat{\mathbf{j}}\hat{P} = \hat{\mathbf{j}}$$

Properties of P and T-operators

Assuming that the wave function is scalar quantity, and that $\hat{P}$ is linear unitary operator we have

$$\hat{P}\psi(\mathbf{r},t) = \psi(-\mathbf{r},t)$$

$$\hat{P} + \hat{\mathbf{r}}\hat{\mathbf{p}}\hat{P} = \left(\hat{P} + \hat{\mathbf{r}}\hat{P}\right)\left(\hat{P} + \hat{\mathbf{p}}\hat{P}\right)$$

$$\hat{P} + \hat{\mathbf{r}}^2\hat{\mathbf{p}}\hat{P} = \left(\hat{P} + \hat{\mathbf{r}}\hat{P}\right)\left(\hat{P} + \hat{\mathbf{r}}\hat{P}\right)\left(\hat{P} + \hat{\mathbf{p}}\hat{P}\right), \quad \ldots\ldots\ldots\ldots$$

$$\hat{P} + \hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t)\hat{P} = \hat{H}\left(\hat{P} + \hat{\mathbf{p}}\hat{P}\,, \hat{P} + \hat{\mathbf{r}}\hat{P}, t\right) = \hat{H}(-\hat{\mathbf{p}},-\hat{\mathbf{r}},t,)$$

Properties of P and T-operators

$$\hat{T}\psi(\mathbf{r},t) = \psi^*(\mathbf{r},-t)$$

$$\hat{T}^+\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t)\hat{T} = H^*\left(\hat{T}^+\hat{\mathbf{p}}\hat{T},\hat{T}^+\hat{\mathbf{r}}\hat{T},t\right) = H^*\,(-\hat{\mathbf{p}},\hat{\mathbf{r}},t)$$

$$\hat{P}^+\hat{T}^+\hat{H}(\hat{\mathbf{p}},\hat{\mathbf{r}},t)\hat{P}\hat{T} = H^*\left(\hat{P}^+\hat{T}^+\hat{\mathbf{p}}\hat{P}\hat{T},\hat{P}^+\hat{T}^+\hat{\mathbf{r}}\hat{P}\hat{T},t\right) = H^*\,(-\hat{\mathbf{p}},\hat{\mathbf{r}},t) =$$

$$= H^*\,(\hat{\mathbf{p}},-\hat{\mathbf{r}},-t)$$

Examples of PT-symmetric systems

$$H = p^2 + x^{2K}(ix)^{\epsilon}$$

$$\mathcal{L} = \frac{1}{2}\,(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + g\phi^2(i\phi)^{\epsilon} \qquad (\epsilon \geq 0)$$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2\,\frac{1}{2}i\bar{\psi}\partial\psi + \frac{1}{2}S'(\phi)\bar{\psi}\psi + \frac{1}{2}[S(\phi)]^2 =$$

$$= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}i\bar{\psi}\partial\psi + \frac{1}{2}g(1+\epsilon)(i\phi)^{\epsilon}\bar{\psi}\psi - \frac{1}{2}g^2(i\phi)^{2+2\epsilon}$$

PT-Symmetric inner product

$$(f,g) = \int dx[PTf(x)]g(x)$$

$$\int dxg(x)[PTHf(x)] = \int dxHg(x)[PTf(x)]$$

Introducing of PT-Symmetry in a quantum system

Similarly to that in Hermitian quantum mechanics, PT-symmetry in a quantum system can be introduced either via the **boundary conditions**, or **complex PT-symmetric potential.**

PT-symmetry in optics

Maxwell's equations reduce to the scalar Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{\omega}{c}\right)^2 \varepsilon(x,z)\right)E(x,z)=0$$

It formally coincides with the stationary Schrodinger equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi_k(x,z)-\frac{2m(V(x,z)-E_k)}{\hbar^2}\psi_k(x,z)=0$$

PT-symmetry in optics

Optical analog of the potential energy in quantum mechanics is the permittivity in optics: PT-symmetry condition for the optical system is defined as the condition imposed on the permittivity of the medium

$$\mathrm{Re}\,\varepsilon(\omega, x, z) = \mathrm{Re}\,(\omega, -x, -z)$$

$$\mathrm{Im}\,\varepsilon(\omega, x, z) = -\,\mathrm{Im}\,(\omega, -x, -z)$$

The stationary Schrödinger equation does not include the time dependence, and therefore the time reversal operation $\hat{T}$ is equivalent conjugation $\hat{K}$.

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nature physics

Observation of parity–time symmetry in optics

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One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables[1]. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity–time (*PT*) symmetry[2–7]. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories[8], non-Hermitian Anderson models[9] and open quantum systems[10,11], to mention a few. Although the impact of *PT* symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where *PT*-related notions can be implemented and experimentally investigated[12–15]. In this letter we report the first observation of the behaviour of a *PT* optical coupled system that judiciously involves a complex index potential. We observe both spontaneous *PT* symmetry breaking and power oscillations violating left-right symmetry. Our results may pave the way towards a new class of *PT*-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and tailored transverse energy flow.

($\varepsilon > \varepsilon_{th}$), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous *PT* symmetry-breaking, that is, a 'phase transition' from the exact to broken-*PT* phase[7,20].

In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in *PT*-symmetric complex potentials. In fact, such *PT* 'optical potentials' can be realized through a judicious inclusion of index guiding and gain/loss regions[7,12–14]. Given that the complex refractive-index distribution $n(x) = n_R(x) + in_I(x)$ plays the role of an optical potential, we can then design a *PT*-symmetric system by satisfying the conditions $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$.

In other words, the refractive-index profile must be an even function of position x whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope E of the optical beam is governed by the paraxial equation of diffraction[13]:

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0[n_R(x) + in_I(x)]E = 0$$

PT-symmetric quantum graph

Skew-Hermitian product on graph, which is defined for arbitrary differential operator, H as

$$\Omega(\psi,\phi) = \langle H\psi,\phi\rangle - \langle\psi,H\phi\rangle$$

$$\Omega(\psi,\phi) = -\sum_j^N \left[\phi^*{}_j(0)\frac{d\psi_j(0)}{dx} - \psi_j(0)\frac{d\phi^*{}_j(0)}{dx}\right] +$$

$$+\sum_j^N \left[\phi^*{}_j(L)\frac{d\psi_j(L)}{dx} - \psi_j(L)\frac{d\phi^*{}_j(L)}{dx}\right] = 0$$

Boundary conditions I

$$\psi_1(0) = \psi_2(0) = \psi_3(0),$$

$$\left.\frac{\partial \psi_1}{\partial x}\right|_{x=L_1} + \left.\frac{\partial \psi_2}{\partial x}\right|_{x=L_2} + \left.\frac{\partial \psi_3}{\partial x}\right|_{x=L_3} = 0,$$

$$\psi_j(L_j) = 0, j = 1, 2, 3.$$

Boundary conditions II

$$\left.\frac{\partial \psi_1}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_2}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_3}{\partial x}\right|_{x=0},$$

$$\psi_1(L_1) + \psi_2(L_2) + \psi_3(L_3) = 0,$$

$$\left.\frac{\partial \psi_j}{\partial x}\right|_{x=L_j} = 0, \quad j = 1, 2, 3.$$

D. U. Matrasulov, K. K. Sabirov, J. R. Yusupov, J. Phys. A **52** 155302 (2019)

PT-symmetric quantum graph

Secular equation for finding energy spectrum

$$e^{ikL_1}\left(1-e^{2ikL_2}\right)\left(1-e^{2ikL_3}\right)+e^{ikL_2}\left(1-e^{2ikL_1}\right)\left(1-e^{2ikL_3}\right)+$$

$$+e^{ikL_3}\left(1-e^{2ikL_1}\right)\left(1-e^{2ikL_2}\right)=0$$

$$\psi_j(x,k_n)=B\,\frac{\sin k_n\left(L_j\ -x\right)}{\sin k_nL_j}$$

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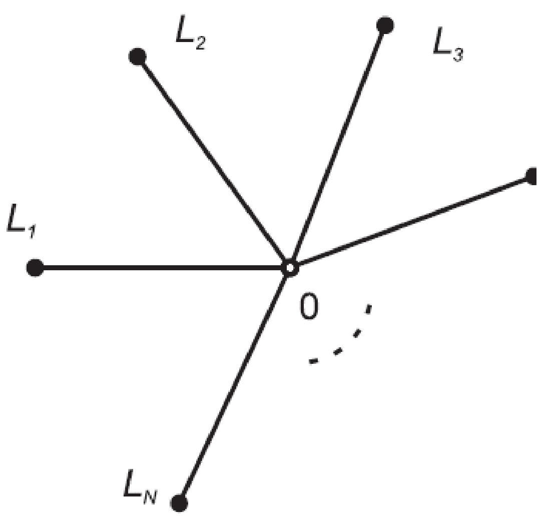
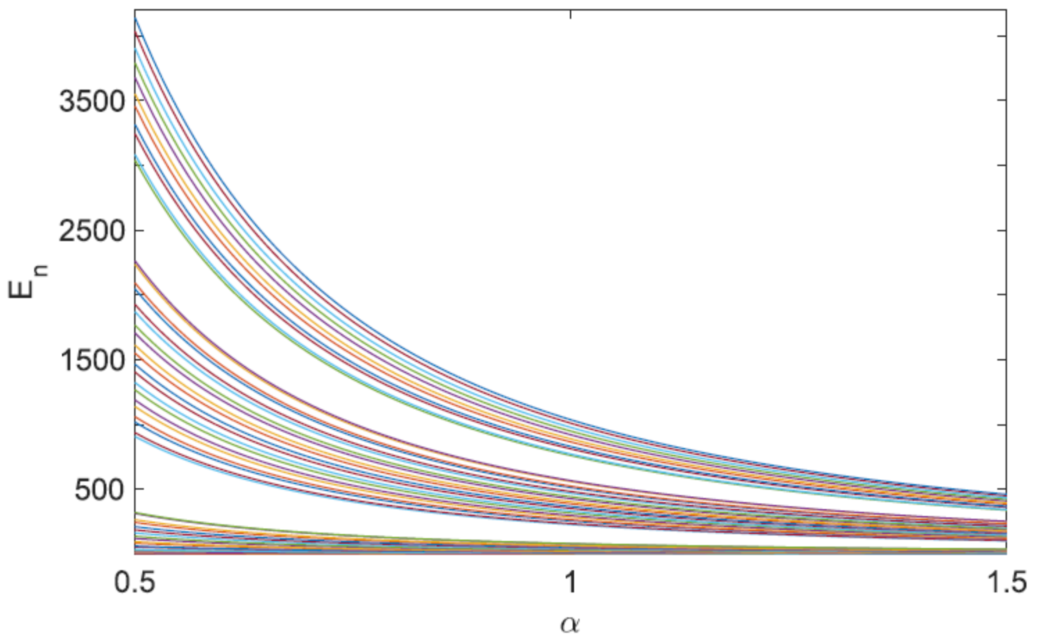
PT-symmetric quantum graphs





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Breaking of Kirchhoff rule

Total current at the vertex (x = 0)

$$J(0,t) = J_1(0,t) + J_2(0,t) + J_3(0,t)$$

$$J_j(0,t) = \frac{i}{2}\left[\psi_j(0,t)\partial_x\psi^*{}_j(0,t) - \partial_x\psi_j(0,t)\psi^*{}_j(0,t)\right]$$

$$\psi_j(x,t) = \sum_n C_n e^{-ik^2{}_n t}\phi_j(x,k_n)$$

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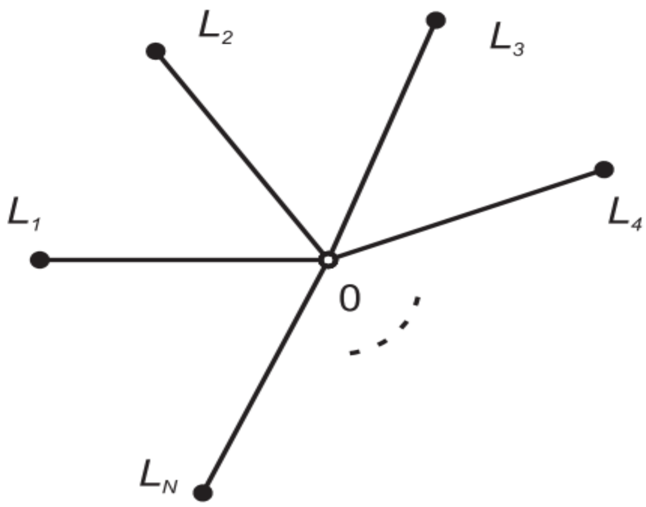
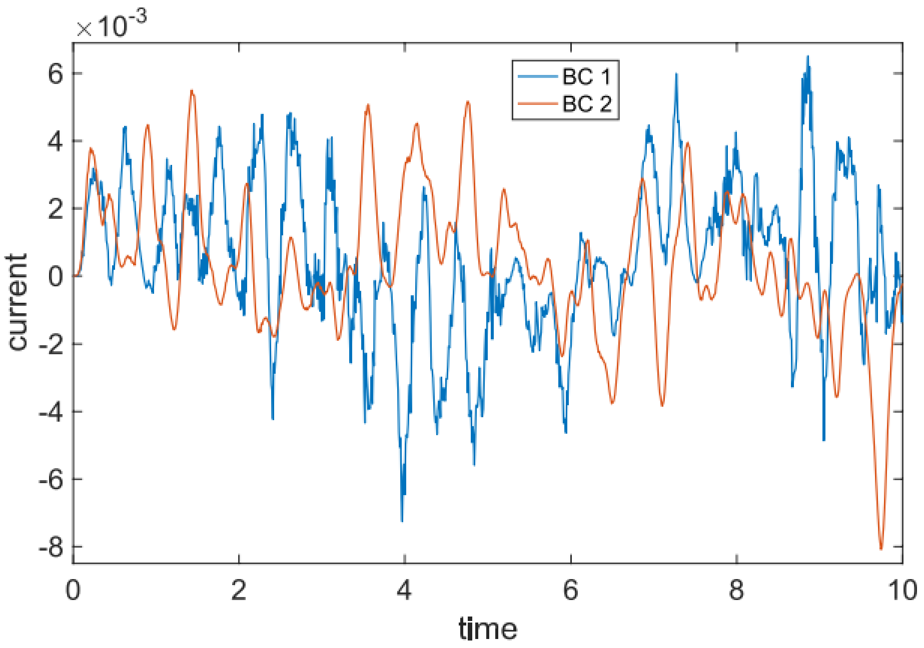
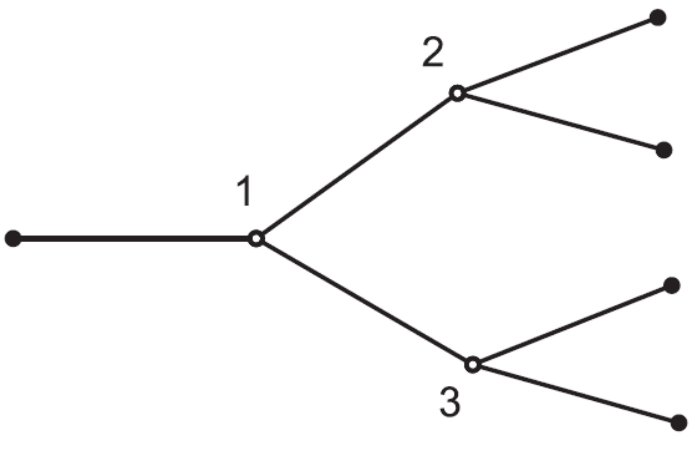
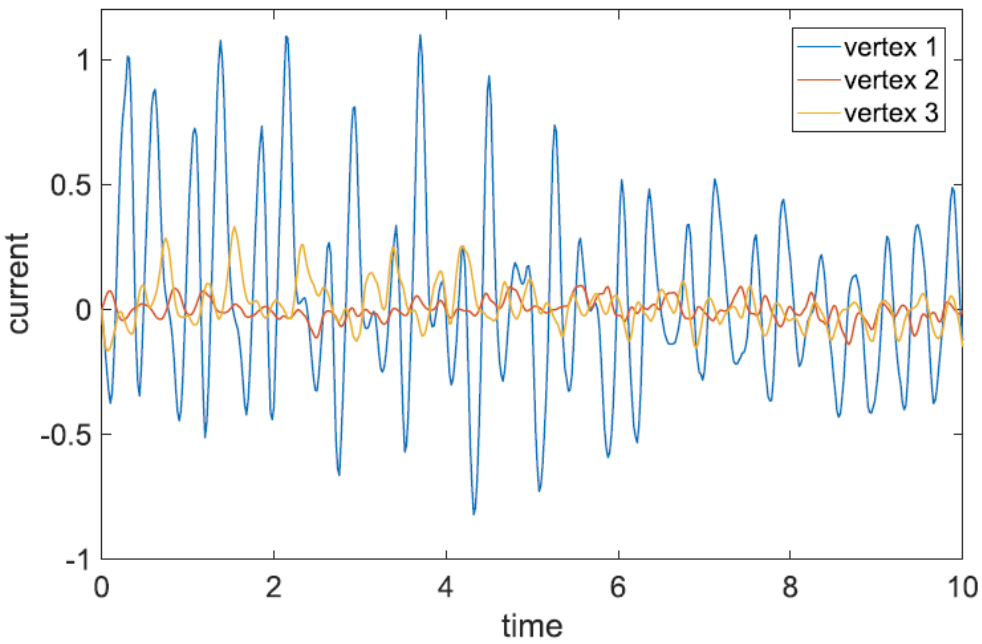
Breaking of Kirchhoff's rule





PT-symmetric quantum graphs





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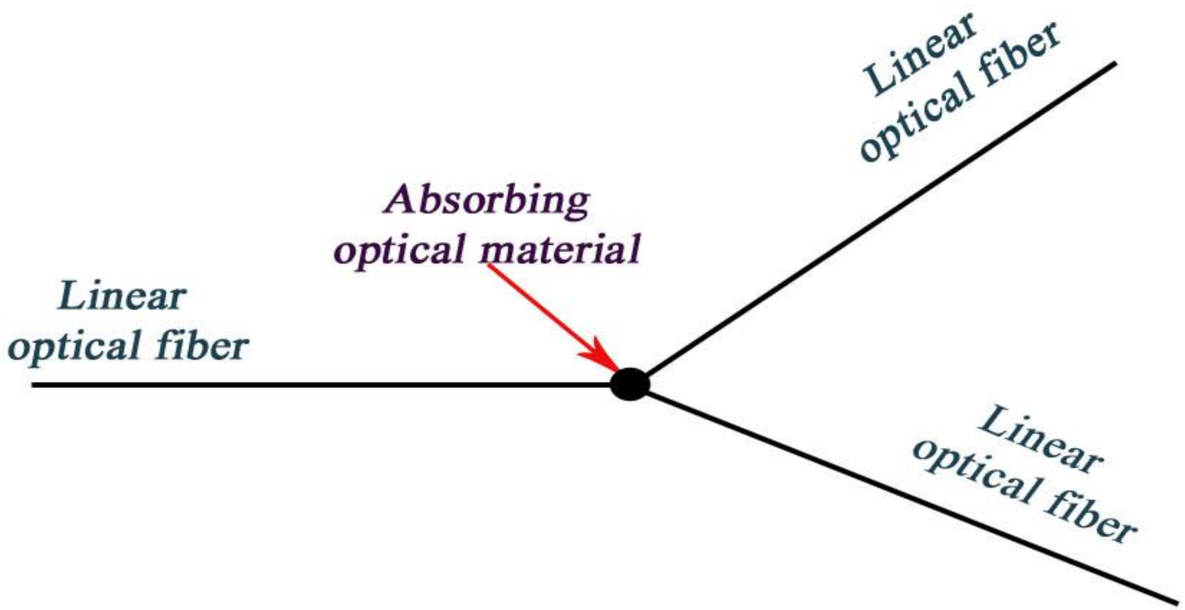
Experimental realization



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Transparent quantum graphs: Reflectionless wave propagation in quantum networks

Absence of backscattering at the graph vertices makes the graph transparent. Mathematically, such transparency can be provided by imposing so-called reflectionless boundary conditions at the graph vertex.

Transparent quantum networks:

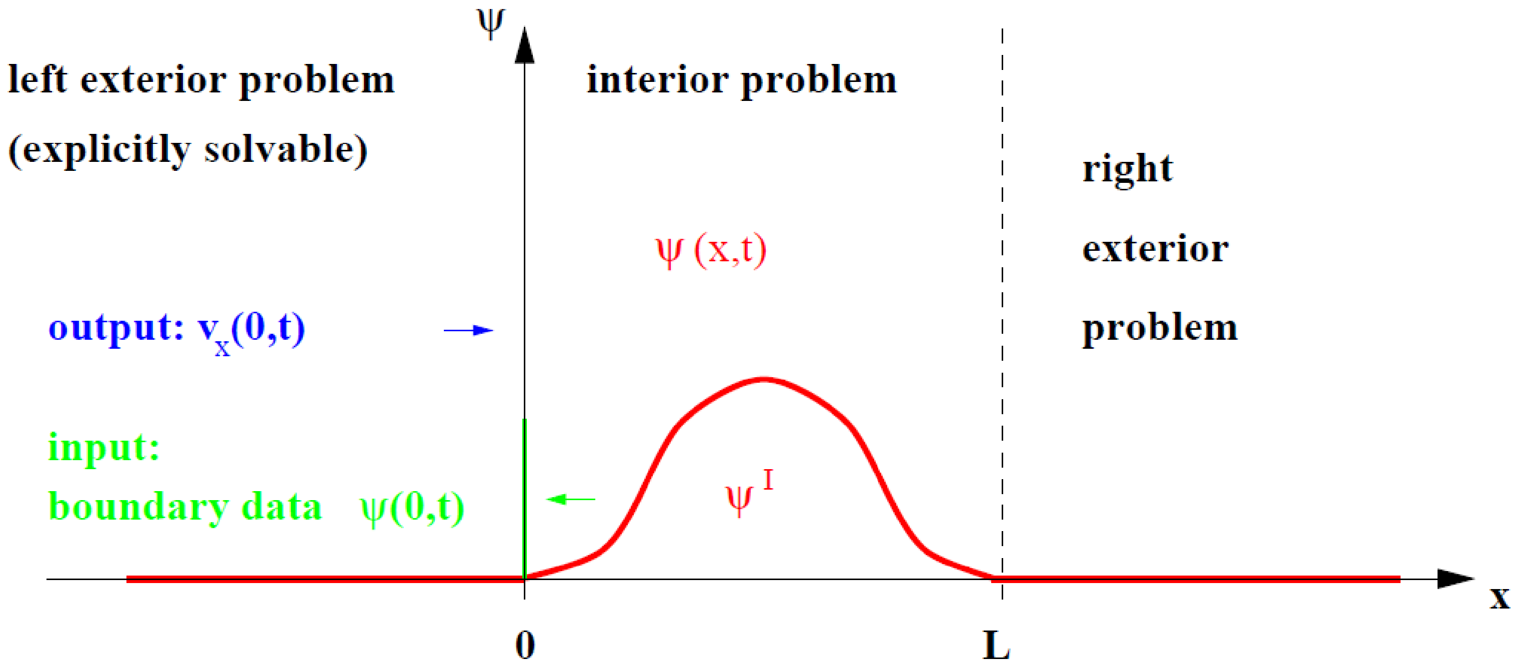


FIGURE 1. Schrödinger equation: Construction idea for transparent boundary conditions

M. Ehrhardt and A. Arnold, Discrete Transparent Boundary Conditions for the Schrödinger Equation, *Rivista di Mathematica della Universita di Parma*, Volume 6, Number 4 (2001), 57-108.

Transparent boundary condition

Interior problem:

$$i\partial_t \Psi = -\frac{1}{2}\partial_x^2 \Psi + V(x,t)\Psi, \quad 0 < x < L, t > 0$$
$$\Psi(x,0) = \Psi^I(x)$$
$$\partial_x \Psi(0,t) = (T_0\Psi)(0,t)$$
$$\partial_x \Psi(L,t) = (T_L\Psi)(L,t)$$

$T_{0,L}$ denote the Dirichlet-to-Neumann maps at the boundaries.

Transparent boundary conditions

$T_{0,L}$ are obtained by solving the two exterior problems:

$$i\partial_t v = -\frac{1}{2}\partial_x^2 v + V_L v, \quad x > L, \quad t > 0$$
$$v(x,0) = 0$$
$$v(L,t) = \Phi(t),\ t > 0,$$
$$\Phi(0) = 0$$
$$v(\infty,t) = 0,$$
$$(T_L\Phi)(t) = \partial_x v(L,t),$$

and analogously for T_0.

Transparent boundary conditions

An inverse Laplace transformation yields the right TBC at $\$x = L\$$:

$$\partial_x \Psi(x = L, t) = -\sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} e^{-iV_L t} \frac{d}{dt} \int_0^t \frac{\Psi(L,\tau) e^{iV_L \tau}}{\sqrt{t-\tau}} d\tau$$

Similarly, the left TBC at $x = 0$ is obtained as

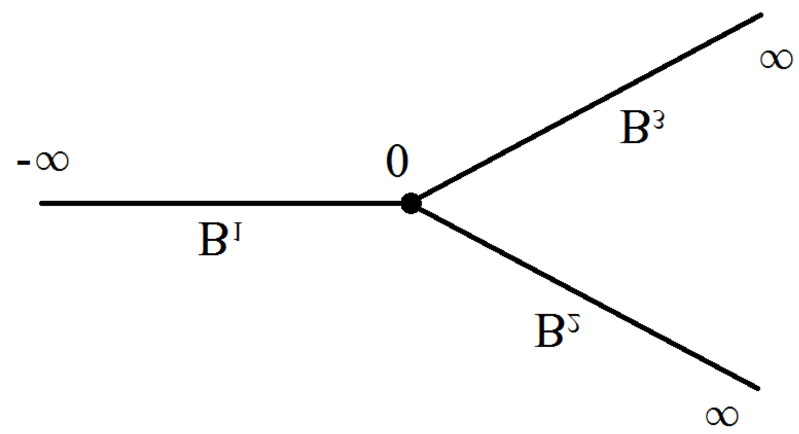
$$\partial_x \Psi(x = 0, t) = -\sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} \frac{d}{dt} \int_0^t \frac{\Psi(L,\tau)}{\sqrt{t-\tau}} d\tau.$$

Transparent quantum networks:

Time-dependent Schrödinger equation for star graph with 3 bonds (in units $\hbar = m = 1$)

$$i\partial_t \Psi_b = -\frac{1}{2}\partial_x^2 \Psi_b, \qquad b = 1,2,3$$

The coordinates assigned to bond B_1 is $x \in (-\infty, 0)$ and $B\_(1,2)$ are $x \in (0, \infty)$.



Transparent quantum networks:

Interior problem for B_1:

$$i\partial_t \Psi_1 = -\frac{1}{2}\partial_x^2 \Psi_1, \quad x < 0,\ t > 0$$

$$\Psi_1(x, 0) = \Psi^I(x)$$

$$\partial_x \Psi_1(0, t) = (T_+ \Psi_1)(0, t)$$

Transparent quantum networks

Exterior problems for $B_{2,3}$:

$$i\partial_t \Psi_{2,3} = -\frac{1}{2}\partial_x^2 \Psi_{2,3}, \quad x > 0, \quad t > 0$$

$$\Psi_{2,3}(x, 0) = 0$$

$$\Psi_{2,3}(0, t) = \Phi_{2,3}(t), \quad t > 0, \quad \Phi_{2,3}(0) = 0$$

$$(T_+ \Phi_{2,3})(t) = \partial_x \Psi_{2,3}(0, t)$$

J. R. Yusupov, K. K. Sabirov, M. Ehrhardt, and D. U. Matrasulov, Phys. Lett. A **383**, 2382 (2019).

Transparent quantum networks

The Laplace transformed current conservation (at x = 0) takes the form

$$\frac{\partial}{\partial x}\widehat{\Psi}_1 = \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial x}\widehat{\Psi}_2 + \frac{\alpha_1}{\alpha_3}\frac{\partial}{\partial x}\widehat{\Psi}_3$$
$$= -\sqrt[+]{-2is}\,\alpha_1^2\left(\frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}\right)$$

Using the inverse transform we have

$$\frac{\partial}{\partial x}\Psi_1(x = 0, t) = A_1\sqrt{\frac{2}{\pi}}\,e^{-i\frac{\pi}{4}}\frac{d}{dt}\int_0^t \frac{\Psi_1(0,\tau)}{\sqrt{t-\tau}}\tau$$

where $A_1 = \alpha_1^2(\alpha_2^{-2} + \alpha_3^{-2})$.

J. R. Yusupov, K. K. Sabirov, M. Ehrhardt, and D. U. Matrasulov, Phys. Lett. A **383**, 2382 (2019).

Transparent quantum networks

Continuity condition:

$$\alpha_1 \Psi_1(0,t) = \alpha_2 \Psi_2(0,t) = \alpha_3 \Psi_3(0,t)$$

Current conservation condition:

$$\frac{1}{\alpha_1}\partial_x \Psi_1(x=0,t) = \frac{1}{\alpha_2}\partial_x \Psi_2(x=0,t) + \frac{1}{\alpha_3}\partial_x \Psi_3(x=0,t)$$

Condition for transparency the continuity and current conservation:

$$\frac{1}{\alpha_1^2} = \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}.$$

J. R. Yusupov, K. K. Sabirov, M. Ehrhardt, and D. U. Matrasulov, Phys. Lett. A **383**, 2382 (2019).

Summary

Basic theory for particle and wave dynamics in quantum networks is presented.

Theory of PT-symmetric graphs:

Breaking Hermitticity in quantum graphs

<u>Experimental realization in microwave fibers</u>

Relativistic quantum graphs with Dirac and Majorana fermions

Transparent quantum graphs: Reflectionless transmission of waves through the vertices.

Outlook

Quantum teleportation on networks

Entangled quantum networks

Qubits in networks

Relativistic quantum graphs: Dirac and Majorana fermions in networks

Transparent microwave networks