# Quantum entanglement: classification and detection

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Samarkand, September 2019

# Outline

I. Introduction
  - The central role of entanglement in quantum information

II. Separability versus entanglement
  - Bipartite case: pure states
  - Bipartite case: mixed states, entanglement criteria
  - Positive maps and entanglement witnesses
  - Entanglement measures
  - Multipartite entanglement

III. Detection of entanglement
  - Operational and non-operational methods
  - Entanglement witnesses and their local decomposition

IV. Hypergraph states
  - Stabilizer formalism
  - Entanglement properties
  - Violation of Bell inequalities

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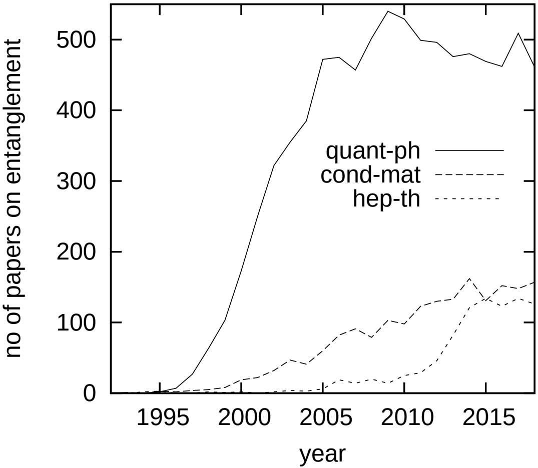
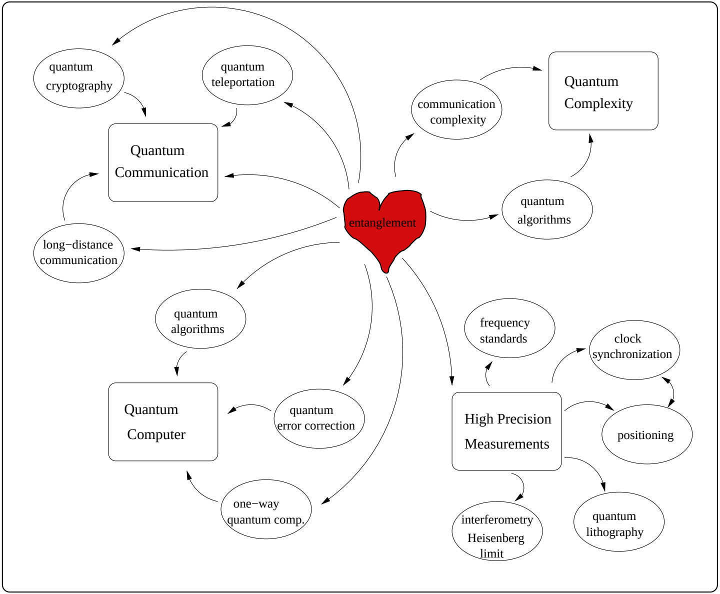


Entanglement is at the heart of quantum information



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The separability problem:

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- example (entangled): Bell states

$$
\begin{aligned}
|\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \\
|\Psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)
\end{aligned}
$$

# Solution of the separability problem for pure bipartite states

Given $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ , $\dim \mathcal{H}_A = d_A$ , $\dim \mathcal{H}_B = d_B \geq d_A$

Pure states:

Schmidt decomposition: $|\psi^r\rangle = \sum_i^r a_i |e_i\rangle |f_i\rangle$ ,

Schmidt rank $r \leq d_A$ , $\langle e_i | e_j \rangle = \delta_{ij} = \langle f_i | f_j \rangle$, $a_i > 0$ , $\sum_i^r a_i^2 = 1$

$|\psi\rangle$ separable $\Leftrightarrow r = 1$

nec. + suff. condition

*Proof:* singular value decomposition
*Note:* $\nexists$ Schmidt decomposition for pure multipartite states i.g.

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$0 \leq p_i \leq 1,\ \sum_i p_i = 1,\ \langle a_i | a_j\rangle \neq \delta_{ij},\ \ \langle b_i | b_j\rangle \neq \delta_{ij}$ in general

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- example (separable):

$$\varrho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

- example (entangled): Werner state

$$\varrho_W = p|\Phi^+\rangle\langle\Phi^+| + (1-p)\frac{1}{4}\mathbb{1} \qquad \text{with} \ \ 1/3 < p \leq 1$$

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separable:

$$\sigma = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

eigenvalues: $\lambda_\sigma = \{\frac{2}{3}, \frac{1}{3}, 0, 0\}$

$$\sigma_A = \sigma_B = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

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entangled:

$$\rho = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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$\rho$ and $\sigma$ have same eigenvalues, also $\rho_A$ and $\sigma_A$ $\hookrightarrow$ conjecture is wrong!

# Operational separability criteria

Given $\varrho$, acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ , $\dim \mathcal{H}_A = d_A$ , $\dim \mathcal{H}_B = d_B \geq d_A$

Mixed states:

Positive partial transpose (PPT) or Peres-Horodecki criterion: $\varrho$ separable $\Rightarrow$ $\varrho^{T_A} \geq 0$

Partial transpose:

$[\varrho^{T_A}]_{m\mu,n\nu} = [\varrho]_{n\mu,m\nu}$ $\hookrightarrow \varrho_{sep}^{T_A} = \sum_i p_i \, \varrho_{A,i}^{T} \otimes \varrho_{B,i}$

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*Proof:*

$\varrho$ separable $\Rightarrow$ $\varrho^{T_A} \geq 0$ *[A. Peres, Phys. Rev. Lett.* **77***, 1413 (1996)]*

$\Leftarrow$ for dim $2 \times 2$ and $2 \times 3$

*[M. + P. + R. Horodecki, Phys. Lett. A* **223***, 1 (1996)]*

*Note:* for dimensions $2 \times 2, 2 \times 3$: nec. + suff. condition

general: nec. condition, $\exists$ PPT ent. states $\equiv$ bound ent. states

# Operational separability criteria – continued

Reduction criterion: $\varrho$ separable $\Rightarrow$

$$\varrho_A \otimes \mathbb{1} - \varrho \geq 0 \quad \text{and} \quad \mathbb{1} \otimes \varrho_B - \varrho \geq 0$$

*[M. Horodecki and P. Horodecki, Phys. Rev. A* **59***, 4206 (1999)]*

*Proof:*
Application of positive map $\Lambda(\sigma) = \mathbb{1} - \sigma$ to B or A

*Note:*
nec. + suff. condition for dimensions $2 \times 2, 2 \times 3$
nec. condition otherwise

# Operational separability criteria – continued

Majorization criterion: $\varrho$ separable $\Rightarrow$

$$\lambda^{\downarrow}_{\varrho_{AB}} \prec \lambda^{\downarrow}_{\varrho_A} \quad \text{and} \quad \lambda^{\downarrow}_{\varrho_{AB}} \prec \lambda^{\downarrow}_{\varrho_B}$$

where: $\lambda^{\downarrow}$ is vector of eigenvalues, in decreasing order;
$x^{\downarrow} \prec y^{\downarrow}$ means $\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow}$ for $k = 1, ..., d-1$

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*Remark:*

in general: $\varrho$ is separable $\Rightarrow$ satisf. PPT $\Rightarrow$ satisf. reduc. crit.
$\Rightarrow$ satisf. major. crit.

in $2 \times 2, 2 \times 3$: $\varrho$ is separable $\Leftrightarrow$ satisf. PPT $\Leftrightarrow$ satisf. reduc. crit.
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# Operational separability criteria – continued

Cross norm ($\equiv$ matrix realignment) criterion:

$\varrho$ separable $\Rightarrow$ $||L(\varrho)|| \leq 1$

$$[L(\varrho)]_{m\mu,n\nu} \quad = \quad [\varrho]_{mn,\mu\nu}$$

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*Proof:* $L$ does not increase *trace norm* of product states:

$$||L(\varrho_A \otimes \varrho_B)|| \leq 1, \quad \text{with} \quad ||A|| = Tr(\sqrt{A^\dagger A})$$

Convexity of norm: $\varrho$ separable $\Rightarrow$ $||L(\varrho)|| \leq 1$

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*Remark:* Complementary to partial transpose criterion: detects some bound entangled states (!), does not detect some free entangled states

# Operational separability criteria – continued

**Local uncertainty relations:**

$\varrho$ separable $\Rightarrow$ $\sum_i \delta^2(M_i) \geq C$

where: $M_i$ local observables, $M_i = A_i \otimes \mathbb{1} + \mathbb{1} \otimes B_i$
Variance of observable $M$: $\delta^2(M)_\varrho = \langle M^2 \rangle_\varrho - \langle M \rangle_\varrho^2$

*[H. Hofmann and S. Takeuchi, Phys. Rev. A 68, 032103 (2003); O. Gühne, Phys. Rev. Lett. 92, 117903 (2004)]*

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*Proof:* Find lower bound $C$ for product states; concavity of variance:

given $\varrho = \sum_k p_k \varrho_k$, $\qquad \sum_i \delta^2(M_i)_\varrho \geq \sum_k p_k \sum_i \delta^2(M_i)_{\varrho_k}$

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*Remark:* Detects some bound entangled states, can detect multipartite entanglement

# Non-operational separability criteria

Positive maps: $\varrho$ is separable $\Leftrightarrow$
for *any* positive map $\Lambda$: $[\mathbb{1}_A \otimes \Lambda_B](\varrho_{AB}) \geq 0$

*[M. Horodecki, P. Horodecki and R. Horodecki; Phys. Lett. A* **223***, 1 (1996)]*

*Definition:* A map $\Lambda$ is positive iff it maps positive operators to positive operators. The map $\Lambda$ is completely positive, iff $\mathbb{1}_d \otimes \Lambda$ is positive.

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*Open problem:* How to characterize non-CP maps?
Only in $d_A d_B \leq 6$ we know $\Lambda_{pos} = \Lambda_{CP}^{(1)} + \Lambda_{CP}^{(2)} \circ T$ $\hookrightarrow$ proof PPT

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Entanglement witnesses:

$\varrho$ is entangled $\Leftrightarrow \exists$ Hermitian operator $\mathcal{W}$ with

$$\begin{aligned} \mathrm{Tr}(\mathcal{W}\varrho) &< 0\,, \\ \mathrm{Tr}(\mathcal{W}\varrho_{sep}) &\geq 0\,. \end{aligned}$$

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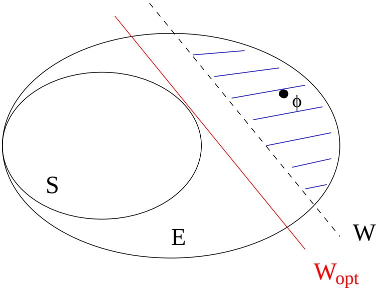

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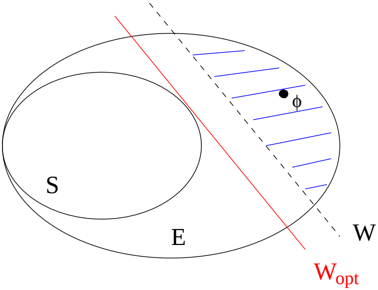


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* $S$ and set of all states are convex, compact; Hahn-Banach theorem

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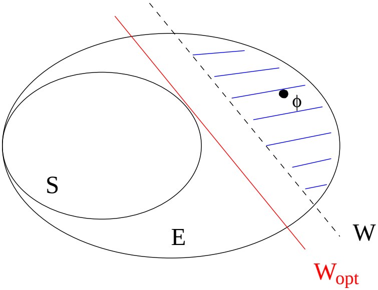


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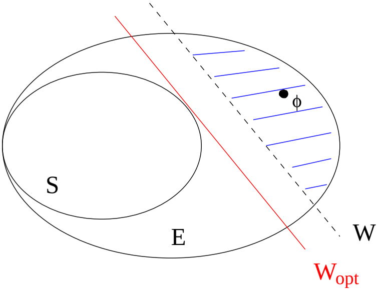


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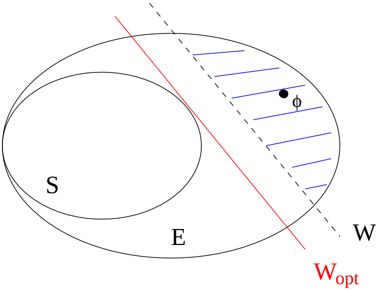


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* $\mathcal{W}$ can be optimized
* *Problem:* need $\infty$ many $\mathcal{W}$

# Example for construction of entanglement witness

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$$x = \max_{|\varphi_{sep}\rangle} |\langle\psi|\varphi_{sep}\rangle|^2 < 1$$

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is an *entanglement witness* that detects $|\psi\rangle\langle\psi|$:

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# Example for construction of entanglement witness

Given an entangled pure state $|\psi\rangle$:

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*Note:* $\mathcal{W}$ also detects $|\psi\rangle\langle\psi|$ plus some noise

# Entanglement measures

Requirements for entanglement measure $E$:

1) $\varrho$ separable $\Rightarrow E(\varrho) = 0$
2) Normalization:

$$E(P_+^d) = \log d$$

3) No increase under LOCC:

$$E(\Lambda_{LOCC}(\varrho)) \leq E(\varrho)$$

4) Continuity:

$$E(\varrho) - E(\sigma) \to 0 \quad \text{for} \quad ||\varrho - \sigma|| \to 0$$

5) Additivity:

$$E(\varrho^{\otimes n}) = n\, E(\varrho)$$

6) Subadditivity:

$$E(\varrho \otimes \sigma) \leq E(\varrho) + E(\sigma)$$

7) Convexity:

$$E(\lambda\varrho + (1-\lambda)\sigma) \leq \lambda E(\varrho) + (1-\lambda)E(\sigma)$$

# Some important entanglement measures

Entanglement cost:

$$E_C(\varrho) = \inf_{\{\Lambda_{LOCC}\}} \lim_{n_\varrho \to \infty} \frac{n^{in}_{|\Phi^+\rangle}}{n^{out}_\varrho}$$

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Distillable entanglement:

$$E_D(\varrho) = \sup_{\{\Lambda_{LOCC}\}} \lim_{n_\varrho \to \infty} \frac{n^{out}_{|\Phi^+\rangle}}{n^{in}_{\varrho}}$$

# Relations and properties of entanglement measures

$$E_D(\varrho) \leq E(\varrho) \leq E_C(\varrho)$$

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Properties of entanglement measures:

| | $E_C$ | $E_F$ | $E_R$ | $E_D$ |
|---|---|---|---|---|
| continuity | ? | $\surd$ | $\surd$ | ? |
| additivity | $\surd$ | no[c] | no[a] | $\surd$ |
| convexity | $\surd$ | $\surd$ | $\surd$ | no (?)[b] |

[a] K. Vollbrecht and R. Werner; quant-ph/0010095

[b] P. Shor, J. Smolin and B. Terhal; Phys. Rev. Lett. 86, 2681 (2001)

[c] M. B. Hastings; Nature Physics 5, 255 (2009)

# Multipartite entanglement (*pure state*, $n$ subsystems)

$|\psi\rangle$ is *fully separable ($n$-separable)* iff

$$|\psi\rangle = \underbrace{|a\rangle \otimes |b\rangle \otimes \ldots \otimes |z\rangle}_{n}$$

$|\psi\rangle$ is *k-separable* w.r.t. specific partition, iff

$$|\psi\rangle = \underbrace{|\alpha\rangle \otimes |\beta\rangle \otimes \ldots \otimes |\omega\rangle}_{\text{k subsystems with dim } d_\alpha, d_\beta, \ldots}$$

$|\psi\rangle$ is *bi-separable* w.r.t. specific partition, iff k=2, i.e.

$$|\psi\rangle = \underbrace{\underbrace{|A\rangle}_{\text{m parties}} \otimes \underbrace{|B\rangle}_{\text{n-m parties}}}_{\text{2 subsystems}}$$

$|\psi\rangle$ is *multipartite entangled* iff it is not bi-separable w.r.t. any bipartition

# Multipartite entanglement (*mixed state*, $n$ subsystems)

$\varrho$ is *fully separable ($n$-separable)* iff (with $p_i \geq 0$ and $\sum_i p_i = 1$)

$$\varrho = \sum_i p_i \left|a_i\right\rangle \left\langle a_i\right| \otimes \left|b_i\right\rangle \left\langle b_i\right| \otimes \ldots \otimes \left|z_i\right\rangle \left\langle z_i\right|$$

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$\varrho$ is *multipartite entangled* iff it is not bi-separable

# Multipartite entanglement witnesses

**Entanglement witness:**

$\varrho$ is entangled $\Leftrightarrow \exists$ Hermitian operator $\mathcal{W}$ with

$$\begin{aligned} \mathrm{Tr}(\mathcal{W}\varrho) &< 0\,, \\ \mathrm{Tr}(\mathcal{W}\varrho_{sep}) &\geq 0\,. \end{aligned}$$

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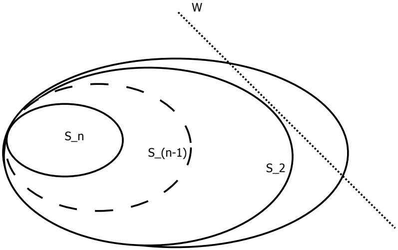

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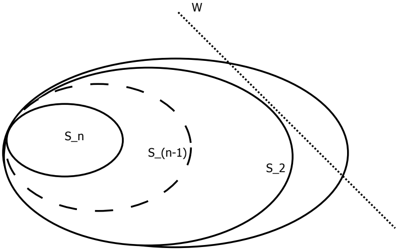


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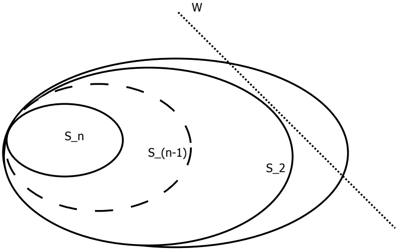


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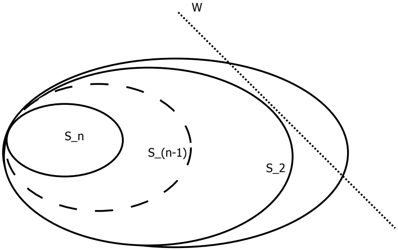


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* how to measure $\mathcal{W}$?

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4. Entanglement witnesses:
   Measure an expectation value, is genuine "quantum" concept
   *Disadvantage:* need to know "something" about state, but this usually holds for experiment

# Local entanglement witnesses

# Local entanglement witnesses

*Theory: [O. Gühne et al, Phys. Rev. A 66, 062305 (2002)]*

- Local decomposition of non-positive witness operator $\mathcal{W}$, pseudo mixture (at least one coeffcient $c_i < 0$):

$$\mathcal{W} = \sum_i c_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i| \ , \quad \text{with} \quad c_i \in \mathbb{R} \ , \quad \sum_i c_i = 1$$

*Experiment, bipartite case: [M. Barbieri et al, Phys. Rev. Lett. 91, 227901 (2003)]*

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- Optimal decomposition of witness? (min. number of meas. settings)
- Signature for entanglement:

$$\begin{aligned} Tr(\mathcal{W}\varrho) \;\; &< \;\; 0 \;\; \Rightarrow \;\; \text{Entanglement} \\ Tr(\mathcal{W}\varrho) \;\; &\geq \;\; 0 \;\; \Rightarrow \;\; ?? \end{aligned}$$

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# (Selected) literature on entanglement

- *"Separability and distillability in composite quantum systems – a primer"*, M. Lewenstein, D. Bruß, J. I. Cirac, B. Kraus, M. Kuś, J. Samsonowicz, A. Sanpera and R. Tarrach, J. Mod. Opt. 47, 2841 (2000)
- *"Characterizing entanglement"*, D. Bruß, J. Math. Phys. 43, 4237 (2002)
- *"Quantum entanglement"*, Ryszard Horodecki, Pawel Horodecki, Michal Horodecki, and Karol Horodecki, Rev. Mod. Phys. 81, 865 (2009)
- *"Entanglement detection"*, O. Gühne and G. Tóth, Phys. Rep. 474, 1 (2009)
- *"Quantum Information: From Foundations to Quantum Technology Applications"*, Eds. D. Bruß and G. Leuchs, Wiley-VCH (2nd Edition, 2019)

# IV. Hypergraph states

M. Rossi, M. Huber, DB, and C. Macchiavello, New J. Phys. 15, 113022 (2013)

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- Hypergraph states for quantum error correction?

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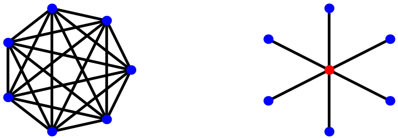

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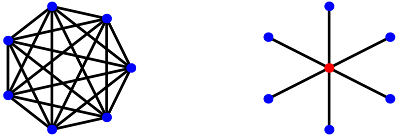
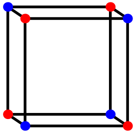


Applications:



- Measurement-based quantum computing
- Quantum error correction (graph codes)
- Quantum metrology
- Multipartite quantum key distribution, secret sharing

# Definition of graph states

Given a graph $G = (V, E)$, i.e. a set of $n$ vertices $V = \{1, ..., n\}$ and a set $E$ of edges $e = \{i, j\}$ with $i, j \in V$.

Corresponding graph state $|G\rangle$:

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- Assign a qubit to each vertex; initial $n$-qubit state given by $|+\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$ with $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- Apply controlled-$Z$ operation $C_e$ for any edge $e$, with $C_e = diag(1, 1, 1, -1)_e = \mathbb{1}_e - 2|11\rangle\langle 11|_e$

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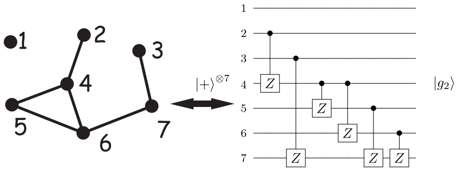


# Graph states: stabilizer formalism

- For any vertex $i$ define operator $g_i$

$$g_i = X_i \otimes Z_{N(i)} = X_i \bigotimes_{j \in N(i)} Z_j$$

where $N(i) = \{j | \{i, j\} \in E\}$ is neighbourhood of vertex $i$, i.e. vertices $j$ which are connected to $i$ by edge

- The graph state $|G\rangle$ is defined via

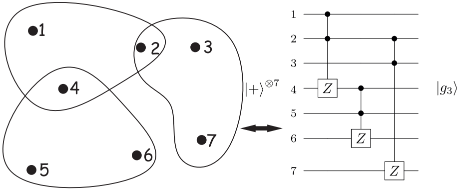
$$g_i |G\rangle = +|G\rangle \quad \text{for all } i = 1, 2, ..., n$$

- The stabilizer operators $\{g_i\}_{i=1,2,...,n}$ generate a commutative group, the stabilizer
- The two definitions of graph states are equivalent

R. Raussendorf, H.-J. Briegel, Phys. Rev. Lett 86, 5188 (2001)

# $k$-uniform hypergraph states

Given a $k$-uniform hypergraph $H^k = (V, E)$, i.e. set of vertices $V = \{1, ..., n\}$ and a set $E$ of hyperedges with cardinality $k$, i.e. $e = \{i_1, ..., i_k\}$ with $i_1, ..., i_k \in V$.



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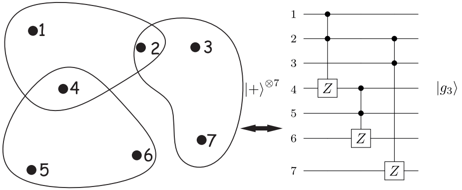


Corresponding $k$-uniform hypergraph state $|H^k\rangle$:

- Assign a qubit to each vertex; i.e. initial state is $|+\rangle^{\otimes n}$
- Apply $k$-qubit controlled $Z$ gate $C_e$ for every $k$-hyperedge:

$$|H^k\rangle = \prod_{e \in E} C_e |+\rangle^{\otimes n}$$

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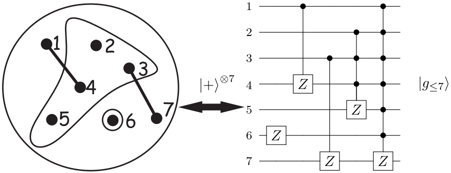

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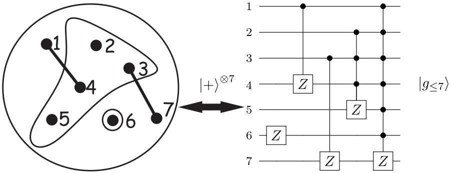


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$$|H\rangle = \prod_{e \in E} C_e |+\rangle^{\otimes n}$$

with $C_e = diag(1, 1, ..., 1, -1) = \mathbb{1}_e - 2|1...1\rangle\langle 1...1|_e$

# Hypergraph states: stabilizer formalism

- For any vertex $i$ define operator $g_i$

$$g_i = X_i \otimes \prod_{e \in E(i)} C_{e \setminus \{i\}}$$

  where $E(i)$ is the set of all edges $e \in E$ with $i \in e$.
  *Note:* these stabilizer operators are "non-local"
  (i.e. no products of Pauli operators)

- The hypergraph state $|H\rangle$ is defined via

$$g_i |H\rangle = +|H\rangle \quad \text{for all } i = 1, 2, ..., n$$

- The stabilizer operators $\{g_i\}_{i=1,2,\ldots,n}$ generate a commutative group, the generalized stabilizer
- The two definitions of hypergraph states are equivalent

M. Rossi, M. Huber, DB, and C. Macchiavello, New J. Phys. 15, 113022 (2013)

# Occurrence of hypergraph states

M. Rossi, DB, and C. Macchiavello, Phys. Rev. A 87, 022331 (2013)

Real Equally Weighted (REW) states for $n$ qubits,
appearing in Deutsch-Josza, Bernstein-Vazirani and Grover's algorithm:

$$|\psi_f\rangle \equiv \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} (-1)^{f(x)} |x\rangle$$

with comp. basis states $|x\rangle \in \{|0...000\rangle, |0...001\rangle, \ldots, |1...110\rangle, |1...111\rangle\}$;

$f(x) : \{0,1\}^n \to \{0,1\}$ Boolean function; $(-1)^{f(x)} = \pm 1$ real phase factor

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*Note:*
Hypergraph states are subset of locally maximally entangleable states

C. Kruszynska and B. Kraus, Phys. Rev. A 79, 052304 (2009)

# Properties of HGS: multipartite entanglement

M. Ghio, D. Malpetti, M. Rossi, DB, and C. Macchiavello, J. Phys. A: Math. Theor. 51, 045302 (2018)

Aim: detect and quantify genuine multipartite entanglement

Def. of multipartite entanglement via smallest bipartite entanglement:

$$E(|\psi\rangle) := \min_{AB} E^{AB}(|\psi\rangle) = 1 - \max_{|\phi^A\rangle|\phi^B\rangle, AB} |\langle\phi^A|\langle\phi^B|\psi\rangle|^2 =: 1 - \alpha(|\psi\rangle)$$

$\hookrightarrow$ find largest Schmidt coefficient $\alpha$ of all bipartite decompositions

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$\hookrightarrow$ find largest Schmidt coefficient $\alpha$ of all bipartite decompositions

Method: Calculate infinity norm for reduced density matrices, where $||M||_\infty$ of $n \times n$ Matrix $M$ is defined as:

$$||M||_\infty := \max_{i=1,2,\ldots,n} \sum_{j=1}^{n} |M_{ij}|$$

As $\lambda_{\max}(M) \leq ||M||_\infty$ for $M \geq 0$, use infinity norm to bound Schmidt coefficients of all bipartitions.

Results: Multipartite entanglement for hypergraph states for $n$ qubits:

Notation: $k$-uniform hypergraph state denoted as $|H_n^k\rangle$

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Further results: Ent. for case of all hyperedges of cardinality $\geq (n-1)$.
Construct entanglement witnesses for detection of multipartite ent.

# Application of HGS: violation of Bell inequalities

Start from Noncontextuality for graph states: remember Mermin inequality for GHZ state (graph state of fully connected graph with 3 vertices)

$$\mathcal{M} = X_1 Z_2 Z_3 + Z_1 X_2 Z_3 + Z_1 Z_2 X_3 - X_1 X_2 X_3$$

If $X_i$ and $Z_i$ are classical quantities with value $\pm 1$, then

$$\langle \mathcal{M} \rangle \leq 2$$

However, interprete $\mathcal{M}$ in terms of stabilizer operators:

$$\mathcal{M} = g_1 + g_2 + g_3 + g_1 g_2 g_3$$

Thus, $\langle \mathcal{M} \rangle = 4$ for GHZ state (eigenstate of $g_i$ with eigenvalue $+1$).
Generalization: find hypergraph states that fulfil

$$\sum_{i=1}^{n} g_i + \prod_{i=1}^{n} g_i = \sum_{i=1}^{n} X_i \prod_{e \in E(i)} C_{e \setminus \{i\}} - \prod_{i=1}^{n} X_i$$

O. Gühne et al, J. Phys. A: Math. Theor. 47, 335303 (2014)

# $k$-uniform hypergraph states fulfilling Mermin argument

Conditions for Mermin argument to hold:

$n$ and $k$ are such that $\binom{n}{k}$ is odd and $\binom{n-s}{k-s}$ is even for any $s < k$

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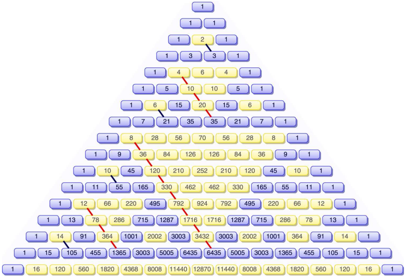


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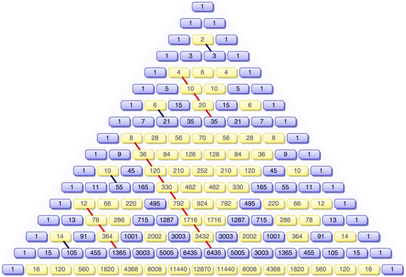


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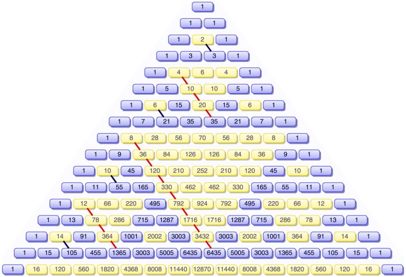


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See also: Exponentially increasing violation of local realism

M. Gachechiladze, C. Budroni, and O. Gühne, Phys. Rev. Lett. 116, 070401 (2016)

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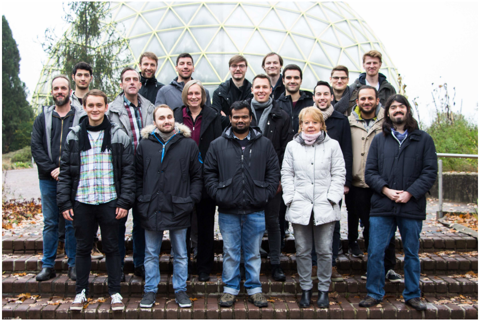
- Entanglement of low-dimensional bipartite quantum systems is well-understood.
- Entanglement of high-dimensional (bipartite or multipartite) quantum systems is not yet fully understood.
- Entanglement can be detected "easily" with present-day technology (local entanglement witnesses).
- There are many interesting families of multipartite entangled states, one example: hypergraph states.

# Quantum Information Theory in Düsseldorf

*Institut für Theoretische Physik III, Heinrich-Heine-Universität Düsseldorf, Germany*



from left to right: J. Bremer, J. M. Henning, D. Miller, H. Kampermann, T. Holz, G. Gianfelici, M. Zibull, DB, T. Backhausen, S. Datta, F. Bischof, T. Wagner, C. Liorni, C. Glowacki, F. Grasselli, C. Hoffmeister, B. Sanvee, L. Tendick, M. Battiato

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